UNCERTAINTY PRINCIPLE FOR RATIONAL FUNCTIONS IN HARDY SPACES

Dan Xiong, Li Chai*
Engineering Research Center of Metallurgical Automation and Measurement Technology
Wuhan University of Science and Technology
Wuhan, China
{xiongdan, chaili}@wust.edu.cn

Jingxin Zhang
School of Software & Electrical Eng.
Swinburne University of Technology
Melbourne, Australia.
jingxinzhang@swin.edu.au

ABSTRACT

Uncertainty principles in finite dimensional vector space have been studied extensively, however they cannot be applied to sparse representation of rational functions. This paper considers the sparse representation for a rational function under a pair of orthonormal rational function bases. We prove the uncertainty principle concerning pairs of compressible representation of rational functions in the infinite dimensional function space. The uniqueness of compressible representation using such pairs is provided as a direct consequence of uncertainty principle.

Index Terms—uncertainty principle, rational functions, orthonormal rational function basis, infinite dimensional function space, sparse representation

1. INTRODUCTION

Uncertainty principles generally express the impossibility for a function (or a vector in the discrete case) to be simultaneously sharply concentrated in two different representations, provided the latter are incoherent enough [1]. Its first statement in quantum mechanics was the so-called Robertson-Schrödinger inequality, which established a lower bound for the product of variances of any two self-adjoint operators on a generic Hilbert space. For various purposes, many such settings using different measures of "concentration" have been proposed in the literature. For example, the uncertainty principles in the setting of discrete sequences, continuous and discrete-time functions for $L_2$ and $L_1$ measures were presented in [2]. Uncertainty principles on the unit sphere have been established in [3]. A very general uncertainty principle for operators on Banach spaces was given in [4] and, in more abstract settings, on compact Riemannian manifolds [5]. The uncertainty principle from an information theory point of view was discussed on compact Riemannian manifolds [5]. The uncertainty principle from an information theory point of view was discussed on compact Riemannian manifolds [5]. The uncertainty principle from an information theory point of view was discussed on compact Riemannian manifolds [5].

Different the representations, the more constraining the bounds. The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.

The direct consequence of uncertainty principle is the uniqueness of sparse representation of the signal $S$ in pair of two bases. The mutual coherence measures how different two representations are. The more different the representations, the more constraining the bounds.
The uncertainty principle in finite dimensional settings is that if two orthonormal bases are mutually incoherent, no nonzero signal can have a sparse representation in both bases simultaneously. As analyzed in [13] and [16], the basic uncertainty principle concerning pairs of representations of finite dimensional vectors in different orthonormal bases has a direct impact on the uniqueness property of the sparse representation of such vectors using pairs of orthonormal bases as overcomplete dictionaries. Hence the uncertainty principle and uniqueness are fundamentally instrumental to the sparse representation of signals under the pairs of orthonormal bases.

It should be noted that uncertainty principle in finite dimensional vector space cannot work for the sparse representation and reconstruction of rational functions which have infinite impulse response. In this paper, we establish analogous uncertainty principles for rational functions represented by orthonormal rational function (ORF) basis, which is a new version for infinite dimensional function space. Orthonormal rational functions have been widely studied over the last twenty years, since the rationality improves efficiency of the representation of linear systems, and orthonormality leads to a great simplification of the analysis and synthesis involved in using the basis functions, see for instance [17–24]. On the basis of the uncertainty principle for rational function, we investigate the sparse representation of a rational function under a pair of ORF bases.

The contributions of this paper are: the proposal of a new definition of sparsity for infinite sequence, and establishment of the uncertainty principle and uniqueness of sparse representation for rational transfer functions, using the uniform bound of maximal absolute inner product of the pair of the ORFs as an index, which extends the results of [13] to infinite dimensional function space and compressible representation.

The rest of this paper is organized as follows. In Section II, the uncertainty principle for sparse representation of rational functions is given. The uniqueness of the representation of rational functions using two ORF bases is presented in Section III. Section IV concludes the paper and discusses the future work. Given the space limitations, we only give the proof outlines of the theorems, see [25] for the proofs in the full version.

### 2. Uncertainty Principle for Sparse Representation of Rational Functions

Rational functions, which are widely used in signal processing and control to model both signals and dynamic systems, can be put in a Hilbert space framework. The Hardy space $H_2$ is a Hilbert space with the inner product between two rational functions $F(z)$ and $G(z)$ defined as

$$
(F(z), G(z)) = \frac{1}{2\pi i} \oint_{\mathbb{T}} F(z) \overline{G(z)} \frac{dz}{z} = \frac{1}{2\pi} \int_{0}^{2\pi} F(e^{i\omega}) \overline{G(e^{i\omega})} d\omega,
$$

where $\mathbb{T} = \{z | |z| = 1\}$.

Given a rational function $H(z) \in H_2$, it has a unique representation in every ORF basis of this space. If $\{\phi_k(z)\}_{k=1}^{\infty}$ is an ORF basis, then we have $H(z) = \sum_{k=1}^{\infty} \alpha_k \phi_k(z)$, where $\alpha_k = \langle H(z), \phi_k(z) \rangle$.

Suppose we have two different ORF bases $\{\phi_k(z)\}_{k=1}^{\infty}$, $\{\psi_l(z)\}_{l=1}^{\infty}$. Then every rational function $H(z)$ has a unique representation under the two bases, respectively, denoted as

$$
H(z) = \sum_{k=1}^{\infty} \alpha_k \phi_k(z) = \sum_{l=1}^{\infty} \beta_l \psi_l(z).
$$

Obviously, if $H(z)$ has a stable pole, then the representation of $H(z)$ using impulse response is infinite, which requires a large number of sampling data to guarantee the approximation performance. While if the pole of the selected rational bases is exactly the same as that of $H(z)$, then the representation will be much shorter. A natural idea is to get a much sparser representation of $H(z)$ in a joint, overcomplete set of ORF bases, say

$$\{\Phi(z), \Psi(z)\} = \{\phi_1(z), \phi_2(z), \ldots, \psi_1(z), \psi_2(z), \ldots\}.$$

Notice that rational functions have infinite impulse coefficients, the sparsity defined in [13] is no longer applicable. We first present a new definition of sparsity, called $\varepsilon$-sparsity and then establish the uncertainty principle for rational functions that leads to the bound yielding uniqueness for the sparse representation of rational function in pairs of ORF bases.

**Definition 1.** For a fixed threshold $\varepsilon > 0$ and an infinite sequence $\alpha = [\alpha_1, \alpha_2, \cdots]^T$ in $l_1$, i.e., $\|\alpha\|_1 = \sum_{k=1}^{\infty} |\alpha_k| < \infty$. Let

$$
N_\varepsilon(\alpha) = \min \{K : \sum_{k=1}^{\infty} |\alpha_k| \leq \varepsilon\}.
$$

The $\varepsilon$-support of $\alpha$ is defined as

$$\Gamma_\varepsilon(\alpha) = \{k : |\alpha_k| \neq 0, 1 \leq k < N_\varepsilon(\alpha)\},$$

and the cardinality of $\Gamma_\varepsilon(\alpha)$ as the $\varepsilon$-0 norm of $\alpha$, denoted by $\|\alpha\|_0(\varepsilon)$.

**Remark 1.** $\alpha$ in $l_1$ guarantees the existence of $N_\varepsilon(\alpha)$. If $\varepsilon = 0$, then $\varepsilon$-support $\Gamma_\varepsilon(\alpha)$ for $\alpha$ is the support of $\alpha$ in the general sense, i.e. $\{k : |\alpha_k| \neq 0\}$.

**Definition 2.** For a given positive integer $s$, the coefficient $\alpha$ is $(\varepsilon, s)$-sparse in the sense of $\varepsilon$-0 norm if $\|\alpha\|_0(\varepsilon) \leq s$. For brevity, we call the coefficient $\alpha$ $\varepsilon$-sparse if the value of $s$ is not concerned.
Definition 3. A rational function is \((\varepsilon, s)\)-sparse if the representation coefficient under an ORF basis is \((\varepsilon, s)\)-sparse.

If \(\varepsilon = 0\), then \(\|\alpha\|_{0(\varepsilon)} = \|\alpha\|_{0}\), which is the number of nonzeros in \(\alpha\), and the \((\varepsilon, s)\)-sparsity becomes the \(s\)-sparsity in compressed sensing. However it should be noted that for a rational function with poles away from zero, its impulse response is usually infinite and cannot be \((0, s)\)-sparse corresponding to \(\varepsilon = 0\), which shows that the definition of the sparsity in traditional compressed sensing is not applicable to the sparsity of the rational function. Hence the uncertainty principle discussed in the sparse representation in dictionaries cannot apply for the sparse representation of rational functions. In the following, based on the \((\varepsilon, s)\)-sparsity, we present the uncertainty principle for the representation of rational functions under two ORF bases.

**Theorem 1.** (Uncertainty Principle) Let \(H(z) \in H_2\) be a rational function that can be represented in (2). For a fixed threshold \(\varepsilon, \|\alpha\|_{0(\varepsilon)}\) and \(\|\beta\|_{0(\varepsilon)}\) are the \(\varepsilon\)-0 norm of \(\alpha\) and \(\beta\), respectively, then for all such pairs of representation we have

\[
\left(\sqrt{\|\alpha\|_{0(\varepsilon)} + \varepsilon}\right)^2 + \left(\sqrt{\|\beta\|_{0(\varepsilon)} + \varepsilon}\right)^2 \geq \frac{2}{\mu},
\]

where

\[
\mu = \sup_k |\langle \phi_k(z), \psi_l(z) \rangle|
\]

and \(\langle \phi_k(z), \psi_l(z) \rangle\) is the inner product of \(\phi_k(z)\) and \(\psi_l(z)\) defined in (1). Following the terminology of compressed sensing, we call \(\mu\) the mutual coherence of two ORF bases \(\{\phi_k(z)\}\) and \(\{\psi_l(z)\}\).

**Proof outline of Theorem 1.** Notice that the transfer function considered is in \(H_2\) space satisfying

\[
H(z) = \sum_{k=1}^{\infty} \alpha_k \phi_k(z) = \sum_{l=1}^{\infty} \beta_l \psi_l(z).
\]

For \(\{\alpha_k\}\) and \(\{\beta_l\}\), denote \(\Gamma_{\varepsilon}(\alpha)\) and \(\Gamma_{\varepsilon}(\beta)\) as the \(\varepsilon\)-support of \(\alpha\) and \(\beta\), respectively. Then we have \(\sum_{k \notin \Gamma_{\varepsilon}(\alpha)} |\alpha_k| \leq \varepsilon\) and \(\sum_{l \notin \Gamma_{\varepsilon}(\beta)} |\beta_l| \leq \varepsilon\).

Without loss of generality, we assume \(\langle H(z), H(z) \rangle = 1\). Note that

\[
1 = \langle H(z), H(z) \rangle = \langle \sum_{k=1}^{\infty} \alpha_k \phi_k(z), \sum_{l=1}^{\infty} \beta_l \psi_l(z) \rangle = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\alpha_k| |\beta_l| |\phi_k(z), \psi_l(z)| \\
\leq \mu \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\alpha_k| |\beta_l| \\
\leq \mu \left( \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k| + \varepsilon \right) \left( \sum_{l \in \Gamma_{\varepsilon}(\beta)} |\beta_l| + \varepsilon \right).
\]

Similarly, we have

\[
1 = \langle H(z), H(z) \rangle = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\alpha_k| |\phi_k(z), \phi_l(z)| |\beta_l| \\
= \sum_{k=1}^{\infty} |\alpha_k|^2 \sum_{l=1}^{\infty} |\alpha_k|^2 |\beta_l|^2 \\
\leq \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k|^2 + \sum_{l \in \Gamma_{\varepsilon}(\beta)} |\beta_l|^2 + \varepsilon^2.
\]

The bound of the above expression can be solved by the optimization problem

\[
\max_{\alpha_k, \beta_l} \left( \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k| + \varepsilon \right) \left( \sum_{l \in \Gamma_{\varepsilon}(\beta)} |\beta_l| + \varepsilon \right)
\]

s.t. \( |\alpha_k| > 0, |\beta_l| > 0\), \( \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k|^2 \geq 1 - \varepsilon^2\), \( \sum_{l \in \Gamma_{\varepsilon}(\beta)} |\beta_l|^2 \geq 1 - \varepsilon^2\).

This can be separated into two optimization problems:

\[
\max_{\alpha_k} \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k| \text{ s.t. } |\alpha_k| > 0, \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k|^2 \geq 1 - \varepsilon^2 \quad (4)
\]

\[
\max_{\beta_l} \sum_{l \in \Gamma_{\varepsilon}(\beta)} |\beta_l| \text{ s.t. } |\beta_l| > 0, \sum_{l \in \Gamma_{\varepsilon}(\beta)} |\beta_l|^2 \geq 1 - \varepsilon^2. \quad (5)
\]

The optimization (4) can be solved by

\[
\max_{\alpha_k} \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k| \text{ s.t. } |\alpha_k| > 0, \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k|^2 = C, \quad (6)
\]

where \(C \in [1 - \varepsilon^2, 1]\) is a constant.

By using Lagrangian multiplier method, we have Lagrangian function

\[
F(|\alpha_k|, \lambda) = \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k| + \lambda \left( \sum_{k \in \Gamma_{\varepsilon}(\alpha)} |\alpha_k|^2 - C \right).
\]

Let the partial differentiation

\[
\frac{\partial F(|\alpha_k|, \lambda)}{\partial |\alpha_k|} = 1 + 2\lambda |\alpha_k| = 0,
\]

Then we have all the \(|\alpha_k|\) are equal when \(k \in \Gamma_{\varepsilon}(\alpha)\).
Denote $\|\alpha\|_0(\varepsilon) = A$. Then $|\alpha_k| = \sqrt{\frac{C}{A}}$. Hence the maxima of (6) is $A$ $\sqrt{\frac{C}{A}}$. Since $C \in [1-\varepsilon^2, 1]$, then the maxima of (4) is $\sqrt{\frac{C}{A}}$.

Similarly, the maxima of (5) is $\sqrt{\frac{\|\beta\|_0(\varepsilon)}{C}}$. Therefore, the maxima of (3) is $(\sqrt{\frac{\|\alpha\|_0(\varepsilon)}{C}} + \varepsilon) \cdot \sqrt{\frac{\|\beta\|_0(\varepsilon)}{C}} + \varepsilon)$, and

$$1 \leq \mu \left( \sum_{k \in \Gamma(\alpha)} |\alpha_k| + \varepsilon \right) \cdot \left( \sum_{l \in \Gamma(\beta)} |\beta_l| + \varepsilon \right) \leq \mu \left( \sqrt{\|\alpha\|_0(\varepsilon) + \varepsilon} \right) \cdot \left( \sqrt{\|\beta\|_0(\varepsilon) + \varepsilon} \right).$$

Using the inequality between the geometric and arithmetic means, we have

$$1 \leq \frac{\mu \left( \sqrt{\|\alpha\|_0(\varepsilon) + \varepsilon} \right)^2 + \left( \sqrt{\|\beta\|_0(\varepsilon) + \varepsilon} \right)^2}{2} \geq \frac{2}{\mu}. \quad \Box$$

3. UNIQUENESS FOR SPARSE REPRESENTATION OF RATIONAL FUNCTIONS USING TWO ORF BASES

Uncertainty principle reveals that in the perspective of $\varepsilon$-0 norm, a rational function $H(z)$ having a sparse representation in the joint set of two ORF bases will have highly nonsparse representation in either of these bases alone. And the uncertainty principle directly determines the bound which guarantees the uniqueness of the sparse representation. Before we present the uniqueness of the sparse representation, we give the definition for the sparse representation in pairs of ORF bases.

**Definition 4.** For a rational function $H(z)$ represented under a pair of orthonormal rational function bases $\{\phi_k(z)\}$ and $\{\psi_l(z)\}$ as

$$H(z) = \sum_{k=1}^{\infty} \theta_k \phi_k(z) + \sum_{l=1}^{\infty} \theta_l \psi_l(z), \quad (7)$$

denote $\theta_1 = [\theta_1^0, \theta_2^0, \ldots]^T$ and $\theta_2 = [\theta_1^1, \theta_2^1, \ldots]^T$, respectively. $H(z)$ is called $(\varepsilon, s)$-sparse if $\|\theta_1\|_0(\varepsilon) + \|\theta_2\|_0(\varepsilon) \leq s$.

**Theorem 2.** (Uniqueness) For $\varepsilon > 0$, assume $H(z)$ is $(\varepsilon, s)$-sparse under a pair of ORF bases $\{\phi_k(z)\}$ and $\{\psi_l(z)\}$. Then the representation (7) is unique if

$$\left( \sqrt{\|\theta_1\|_0(\varepsilon) + \varepsilon} \right)^2 + \left( \sqrt{\|\theta_2\|_0(\varepsilon) + \varepsilon} \right)^2 < \frac{1}{\mu},$$

where $\theta_1$ and $\theta_2$ are as defined in Definition 4.

**Proof outline of Theorem 2.** Suppose there are two different sparse representations of transfer function $H(z)$, that is

$$H(z) = \sum_{k=1}^{\infty} \theta_k^0 \phi_k(z) + \sum_{l=1}^{\infty} \theta_l^0 \psi_l(z)$$

and

$$H(z) = \sum_{k=1}^{\infty} \theta_k^1 \phi_k(z) + \sum_{l=1}^{\infty} \theta_l^1 \psi_l(z)$$

According to the uncertainty principle, we have

$$\left( \sqrt{\|\theta_1 - \xi_1\|_0(\varepsilon) + \varepsilon} \right)^2 + \left( \sqrt{\|\theta_2 - \xi_2\|_0(\varepsilon) + \varepsilon} \right)^2 \geq \frac{2}{\mu} \cdot \quad (8)$$

However, based on the sparsity assumption of Theorem 2

$$\left( \sqrt{\|\theta_1 - \xi_1\|_0(\varepsilon) + \varepsilon} \right)^2 + \left( \sqrt{\|\theta_2 - \xi_2\|_0(\varepsilon) + \varepsilon} \right)^2 < \frac{1}{\mu},$$

which contradicts (8).

**Remark 2.** Theorems 2 shows that there cannot be two different $\theta^T = [\theta_1^T \theta_2^T]$ obeying $\sqrt{\|\theta_1\|_0(\varepsilon) + \varepsilon}^2 + \sqrt{\|\theta_2\|_0(\varepsilon) + \varepsilon}^2 \leq \frac{1}{\mu}$ that represent the same rational function. And if $\varepsilon = 0$ and the sequence is finite, then the results of Theorems 1 and 2 are parallel to the results of [13].

4. CONCLUSION

A novel uncertainty principle for sparse representation of rational functions in infinite dimensional function space is presented in this paper. The bound which guarantees the uniqueness of the sparse representation is presented using mutual coherence as a measure. Since rational functions are widely used to model both signals and dynamic systems, the consequence in this paper can be applicable to sparse representation of signal and systems using pairs of ORF bases.
5. REFERENCES


