CORRENTROPY-BASED ADAPTIVE FILTERING OF NONCIRCULAR COMPLEX DATA

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ABSTRACT

Real world complex-valued signals typically exhibit rotation-dependent distributions (noncircularity), and significant performance gains in learning algorithms can be obtained by accounting for information beyond the standard second-order noncircularity (impropriety). To this end, we introduce a new closed form definition of complex correntropy which is general enough to cater for both circular and noncircular distributions in complex data, and serves as a basis for a novel cost function for widely linear adaptive filtering, termed the maximum improper complex correntropy criterion (MICCC). A stochastic gradient adaptive filtering algorithm is developed based on the MICCC, and its standard and complementary convergence and stability analyses are conducted with respect to both the circularity of the estimation error and the kernel size in the underlying Parzen estimator. Performance advantages over the strictly linear correntropy algorithm (MCCC) and the mean square error based complex least mean square (CLMS) and augmented CLMS (ACLMS) are demonstrated through analysis and simulations.

Index Terms— Complex-valued signal processing, complementary mean square analysis, complex correntropy, impropriety, maximum improper complex correntropy criterion (MICCC).

1. INTRODUCTION

Standard covariance and entropy-based statistical measures either cannot model higher-order statistics within a time series or/and employ a rigid assumption that the signal of interest is at least second-order stationary. However, in practice, most measured quantities exhibit a degree of non-Gaussianity and nonstationarity, which for some problems can be known in advance. To this end, an extension of the fundamental definition of correlation for random processes was proposed in [1], termed the correntropy, to address the problem that most of the conventional information theoretic learning (ITL) measures [2] do not use all the information in the case of temporally correlated (non-white) input signals. Unlike standard correlation, this measure contains higher-order moments of the probability distribution function (pdf), but is much simpler to estimate directly from the samples than conventional moment expansions. The concept was initially introduced for univariate random processes, and was extended to a more general case of two arbitrary random variables in [3].

Recent application studies have validated correntropy as an efficient tool for analyzing higher-order statistical moments in non-Gaussian signals [3, 4, 5]. Especially successful has been its application as a cost function in linear adaptive filters, within the framework called maximum correntropy criterion (MCC) [6]. Tools developed based on this concept include the MCC-based variable step-size least mean square (LMS) algorithm [7] and a closed-form fixed-point recursion filter [8, 9].

More recently, the correntropy framework has been extended to complex-valued time series, through \textit{complex correntropy}, and the corresponding \textit{maximum complex correntropy criterion} (MICCC), the utility of which as a cost function for the complex-valued least-mean squares (CLMS) and complex-valued fixed-point recursion filters was demonstrated in [10].

Although complex correntropy and the MCC cost function have shown significant potential in complex-valued signal processing, there remain several issues that need to be addressed prior to its more widespread application, these include: (i) complex correntropy was derived with the assumption of proper (second-order circular) random variables, however, this is very restrictive as real world signals are typically second-order noncircular (improper); (ii) the existing MCC cost function is only effective for \textit{strictly linear} (SL) models which are inadequate for the widely linear systems, typical in the real world, and the associated impropriety (second-order noncircularity) of data.

To this end, we introduce novel correntropy measures within a widely linear framework [11], and illustrate their effectiveness for robust adaptive filtering of noncircular data. This is achieved based on the augmented complex statistics which employs both the covariance, $R = E\{xx^H\}$, and the pseudo-covariance, $P = E\{xx^T\}$, matrices, in order to cater for both circular (rotation-invariant probability distributed) and second-order noncircular (improper) signals with rotation dependent pdfs. The introduced maximum improper correntropy criterion (MICCC) is used as a cost function for a corresponding general widely linear adaptive filter. Its convergence is analysed in terms of the mean and mean-square convergence, and uniquely by employing the complementary convergence analysis to assess the degree of circularity of the output error along the iterations. Illustrative simulations demonstrate the MICCC outperforms the MCC-CLMS, CLMS and augmented CLMS (ACLMS).

2. MAXIMUM COMPLEX CORRENTROPY CRITERION

The probabilistic interpretation of complex correntropy is based on estimating the probability of the event $x = y$, for random complex variables $x, y \in \mathbb{C}^N$. This is equivalent to considering the joint probability of the events $\Re\{x\} = \Re\{y\}$ and $\Im\{x\} = \Im\{y\}$ [10]. Using a complex-valued Gaussian pdf, denoted by $\kappa_{\sigma} (\cdot)$, the calculation of correntropy between variables $x$ and $y$ is then equivalent to estimating the probability of the estimation error, $e = x - y$, that is

$$
V_\sigma (x, y) = E\{\kappa_{\sigma} (e)\} = \frac{1}{\pi \sigma^2} \int \exp \left\{- \frac{e^H e}{\sigma^2}\right\}, \hspace{1cm} (1)
$$

whereby $\sigma^2$ is the variance of $\kappa_{\sigma} (e)$ and $V_\sigma (x, y)$ denotes an appropriate Parzen estimator [12], which has the form

$$
V_\sigma (x, y) = \frac{1}{\pi \sigma^2} \frac{1}{N} \sum_{n=1}^{N} \exp \left\{- \frac{|x_n - y_n|^2}{\sigma^2}\right\}, \hspace{1cm} (2)
$$

and represents a measure of maximum similarity between the random variables $x = [x_1, ..., x_N]^T$ and $y = [y_1, ..., y_N]^T$. 

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The “proper” MCCC, proposed in [10], assumes the form in (2) and produces the probabilistic difference, \( e = d - y \), between the desired signal \( d \in \mathbb{C} \) and the filter output \( y \in \mathbb{C} \). In other words, for a strictly linear model \( y = h^H x \), where \( x \in \mathbb{C}^{N} \) is the complex-valued input and \( h \in \mathbb{C}^{N} \) a weight vector, the MCCC represents the maximum complex correntropy between the random variables \( d \) and \( y \), that is
\[ J_{\text{MCCC}} = V_e(d, y) = E\{\kappa_{\sigma, \rho}(e)\}. \tag{3} \]

3. MAXIMUM IMPROPER COMPLEX CORRENTROPY CRITERION

The pdf of a general zero-mean noncircular complex Gaussian distributed random variable, \( x \in \mathbb{C}^{N} \), is defined as [13, 11, 14]
\[ \kappa_{\sigma, \rho}(x) = \frac{1}{\pi\sigma^2 \sqrt{1 - |\rho|^2}} \exp\left\{ \frac{|x|^2 - R\{g(x, x)^2\}}{\sigma^2(1 - |\rho|^2)} \right\}, \tag{4} \]
where \( \rho = E\{x^2\} / E\{|x|^2\} \) is the circularity quotient of \( x \) [15].

To introduce a measure of improper complex correntropy, as an extension of the probabilistic interpretation in [10], consider an improper complex random variable, \( e = x - y \in \mathbb{C}^{N} \), with \( x = [x_1, \ldots, x_N]^T \) and \( y = [y_1, \ldots, y_N]^T \). Then, the complex correntropy is estimated through an appropriate Parzen estimator, given by
\[ V_{\sigma, \rho}(x, y) = \frac{1}{\pi\sigma^2 \sqrt{1 - |\rho|^2}} \sum_{n=1}^{N} \exp\left\{ \frac{e_n^2 - R\{g(e, e)^2\}}{\sigma^2(1 - |\rho|^2)} \right\}, \tag{5} \]
where \( e_n = x_n - y_n \) and \( \rho = E\{e^2\} / E\{e^H e\} \). Well-established methods exist for determining the optimal value for the kernel size, \( \sigma^2 \) in (5) [16, 15], which without loss of generality is assumed to be a constant in this work.

A further insight into the improper complex correntropy is conveniently provided through its Taylor series expansion
\[ V_{\sigma, \rho}(x, y) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} E\left\{ \frac{|e|^2 - R\{g(e, e)^2\}}{\sigma^2(1 - |\rho|^2)^{n+1}} \right\}. \tag{6} \]

Remark 1: Observe that with an increase in the kernel size, \( \sigma^2 \), the higher-order terms in (6) decay faster than the second-order terms, and that, contrary to the case of a proper \( x \), as desired the circularity quotient \( \rho \) is involved too. The larger the circularity coefficient \( |\rho| \), the greater the contribution of the higher-order terms.

Remark 2: The only case where the proposed improper complex correntropy behaves like the covariance is when the kernel size, \( \sigma^2 \), tends to infinity and the circularity quotient, \( \rho \), vanishes. In this way, the involvement of the circularity quotient within the higher-order terms overcomes the undermodeling problem of the proper correntropy model in [10] when applied to noncircular data.

To support the development of correntropy-based adaptive signal processing algorithms for noncircular data, we next introduce the maximum improper complex correntropy criterion (MICCC) which is based on (5), accounts for the complex imprtivity, and can be used for both circular and noncircular inputs.

4. ROBUST WIDELY LINEAR FILTERING

Consider a widely linear (WL) model in the form
\[ y = h^H x + g^H x^* = w^H x, \tag{7} \]
where \( x = [x^T, x^H]^T \) and \( w = [h^T, g^T]^T \) are respectively the augmented input and coefficient vectors, with \( x, h, g \in \mathbb{C}^{N} \) [17]. Define the estimation error, \( e = c - d \), as the difference between the desired signal \( d \in \mathbb{C} \) and the filter output \( y \in \mathbb{C} \). The new cost function is then defined as the maximum improper complex correntropy between the random variables \( d \) and \( y \), and is given by
\[ J_{\text{MICCC}} = V_e(d, y) = E\{\kappa_{\sigma, \rho}(e)\}. \tag{8} \]

4.1. MICCC-based stochastic gradient adaptive filter

Following on the work in [6], we now derive a gradient-based adaptive learning algorithm using the MICCC as a cost function. For an input signal \( x_k = [x_{k-N+1}, \ldots, x_k]^T \) at time instant \( k \), the improper correntropy between the desired signal \( d_k = [d_{k-N+1}, \ldots, d_k]^T \) and the filter output \( y_k = [y_{k-N+1}, \ldots, y_k]^T \) is computed using a sliding window of \( N \) samples, to give
\[ J_k = \frac{1}{\pi\sigma^2 \sqrt{1 - |\rho|^2}} \frac{1}{N} \sum_{i=k-N+1}^{k} \exp\left\{ \frac{e_i^2 - R\{g(e_i^2)^2\}}{\sigma^2(1 - |\rho|^2)} \right\}, \tag{9} \]
where \( e_i = d_i - w_i^H x_i \). The cost function \( J_k \) is maximised with respect to \( w_k \) using gradient ascent [11], that is, based on \( w_{k+1} = w_k + \eta \frac{\partial J_k}{\partial w} \). The computation of the derivative \( \frac{\partial J_k}{\partial w} \) can be simplified through the CR (or Wirtinger) derivative chain rule [18, 11]
\[ \frac{\partial J_k}{\partial w} = \frac{\partial J_k}{\partial e} \frac{\partial e}{\partial w} + \frac{\partial J_k}{\partial \eta} \frac{\partial \eta}{\partial w}. \tag{10} \]

With \( \frac{\partial J_k}{\partial e} = -x \) and \( \frac{\partial J_k}{\partial \eta} = 0 \), equation (10) reduces to
\[ \frac{\partial J_k}{\partial w} = \frac{\partial J_k}{\partial e} \frac{\partial e}{\partial w} = -\frac{\partial \kappa_{\sigma, \rho}(e)}{\partial e} \frac{\partial e}{\partial w} \tag{11} \]
To simplify the derivation of \( \frac{\partial J_k}{\partial w} \), we assume an unbiased estimation with \( E\{e\} = 0 \), such that \( \frac{\partial \kappa_{\sigma, \rho}(e)}{\partial e} = 0 \), to give
\[ \frac{\partial J_k}{\partial w} = E\left\{ \frac{\kappa_{\sigma, \rho}(e)}{\sigma^2(1 - |\rho|^2)} \right\}. \tag{12} \]
Therefore, the weight update for the filter in (7) becomes
\[ w_{k+1} = w_k + \mu \frac{\kappa_{\sigma, \rho}(e_k)}{\sigma^2(1 - |\rho|^2)} x_k \tag{13} \]
The instantaneous approximation \( (N = 1) \) finally yields the weight update of the proposed widely linear correntropy adaptive filter in the form
\[ w_{k+1} = w_k + \mu \frac{\kappa_{\sigma, \rho}(e_k)}{\sigma^2(1 - |\rho|^2)} x_k \tag{14} \]

5. CONVERGENCE ANALYSIS

The mean and mean-square convergence analyses use the following standard independence assumptions:

A1. The desired response is produced by a WL model given by
\[ d_k = h_{\text{opt}}^H x + g_{\text{opt}}^H x^* + \eta_k = w_{\text{opt}}^H x_k + \eta_k, \tag{15} \]
where \( \eta_k \) is complex circular Gaussian noise, that is \( E\{\eta_k^2\} = 0 \), which is uncorrelated with \( x_k \), and \( w_{\text{opt}} \) is the optimal weight vector.

A2. The input \( x_k \) is correlated second-order noncircular such that the off-diagonal elements of \( R_x = E\{x_k x_k^H\} \) and \( P_x = E\{x_k x_k^T\} \) do exist.

A3. The error nonlinearity \( \kappa_{\sigma, \rho}(e_k) \) is asymptotically uncorrelated with \( E\{x_k x_k^H\} \) and \( E\{x_k x_k^T\} \) at the steady state.

A4. The filter is long enough such that the a priori error is zero-mean Gaussian.

The assumption A1 is common, while for a long enough filter the assumption A3 also becomes realistic. Assumption A4 is reasonable owing to the central limit theorem, and also remains valid in the whole adaptation stage [19, 20, 21, 22]. For detailed derivations, we refer to [23].
5.1. Convergence in the mean

Consider the weight error vector, given by \( \mathbf{v}_k = \mathbf{w}_k - \mathbf{w}_{opt} \), so that the estimation error can be expressed in terms of \( \mathbf{v}_k \), as

\[
e_k = d_k - (\mathbf{v}_k + \mathbf{w}_{opt})^H \mathbf{x}_k = \eta_k - \mathbf{v}_k^H \mathbf{x}_k.
\]

For convenience, we introduce the variable

\[
\mathbf{\Pi} = \frac{1}{\mu R_k} \sigma^2 \mathbf{e}_k.
\]

Upon inserting (16)-(17) into (14) we arrive at

\[
\mathbf{v}_{k+1} = \mathbf{v}_k + \mathbf{P}_k \mathbf{R}_k^H \mathbf{y}_k + \mathbf{R}_k \mathbf{Q}_k^H \mathbf{e}_k - \mathbf{P}_k \mathbf{R}_k \mathbf{Q}_k^H \mathbf{e}_k.
\]

Under the convergence assumptions A1-A4, and upon taking the statistical expectations on both sides, we obtain

\[
E \{ \mathbf{v}_{k+1} \} = E \{ \mathbf{v}_k \} + \mathbf{P}_k \mathbf{R}_k^H E \{ \mathbf{y}_k \} - \mathbf{P}_k \mathbf{R}_k \mathbf{Q}_k^H E \{ \mathbf{e}_k \},
\]

where \( \mathbf{R}_k = E \{ \mathbf{x}_k \mathbf{x}_k^H \} \) and \( \mathbf{P}_k = E \{ \mathbf{x}_k \mathbf{x}_k^H \} \) are the covariance and pseudo-covariance matrices of the input data.

At the steady state the terms that include high powers of \( \sigma_k \) can be neglected [24], unless the measurements of the desired signals \( d_k \) are extremely noisy. Further, it is insightful to inspect the first two terms of the Taylor series expansion of \( E \{ \mathbf{v}_k \} \), based on (6), that is

\[
E \{ \mathbf{v}_k \} \approx \frac{1}{\mu R_k} \sigma^2 \left[ 1 + \frac{\sigma^2}{\sigma^2 (1 - |\mathbf{r}|)^2} + \frac{\sigma^2}{\sigma^2 (1 - |\mathbf{r}|)^2} \right] \mathbf{e}_k.
\]

5.2. Convergence in the mean-square

Consider the evolution of the weight error covariance matrix, \( \mathbf{K}_k \), which can be used to determine the MSE through the relation

\[
E \{ |e_k|^2 \} = \sigma^2 + \text{tr} \{ \mathbf{R}_k \mathbf{K}_k \},
\]

where \( \mathbf{R}_k \mathbf{K}_k \) is the excess MSE at time instant \( k \) [25]. The computation of \( E \{ \mathbf{v}_k \mathbf{v}_k^H \} \) is based on (18), and upon taking the expectations of the fourth-order moments using Isserlis’ theorem [26] for Gaussian vectors, we obtain

\[
\mathbf{K}_{k+1} = \mathbf{K}_k + E \{ \mathbf{P}_k \mathbf{R}_k^H \mathbf{y}_k + \mathbf{R}_k \mathbf{Q}_k^H \mathbf{e}_k - \mathbf{P}_k \mathbf{R}_k \mathbf{Q}_k^H \mathbf{e}_k \}
\]

\[
+ E \{ \mathbf{P}_k \mathbf{R}_k^H \mathbf{y}_k + \mathbf{R}_k \mathbf{Q}_k^H \mathbf{e}_k - \mathbf{P}_k \mathbf{R}_k \mathbf{Q}_k^H \mathbf{e}_k \}
\]

\[
- 2 \sigma^2 \mathbf{R}_k \mathbf{G}_k \mathbf{P}_k^* - 2 \sigma^2 \mathbf{R}_k \mathbf{Q}_k \mathbf{R}_k - 2 \mathbf{R}_k \mathbf{K}_k - 2 \mathbf{R}_k \mathbf{K}_k^T - 2 \mathbf{P}_k \mathbf{K}_k^T \mathbf{P}_k^*
\]

\[
+ 2 \mathbf{R}_k \mathbf{G}_k \mathbf{P}_k^* + 2 \mathbf{R}_k \mathbf{Q}_k \mathbf{R}_k + 2 \mathbf{R}_k \mathbf{K}_k - 2 \mathbf{R}_k \mathbf{K}_k^T - 2 \mathbf{P}_k \mathbf{K}_k^T \mathbf{P}_k^*
\]

Next, consider the unitary matrix \( \mathbf{Q} \) derived from the approximate uncorrelating transform (AUT) [27] which diagonalizes the pseudocovariance matrix, \( \mathbf{P} = E \{ \mathbf{x}_k \mathbf{x}_k^T \} \), as \( \mathbf{P} = \mathbf{Q} \mathbf{A}_P \mathbf{Q}^T \), with \( \mathbf{QQ}^T = \mathbf{I} \) and \( \mathbf{A}_P = \text{diag} \{ \lambda_{max}, \ldots, \lambda_{min} \} \) being a diagonal matrix of real-valued entries (circularity coefficients). The AUT also simultaneously approximates diagonalizes the covariance matrix, \( \mathbf{R} = E \{ \mathbf{x}_k \mathbf{x}_k^H \} \), as \( \mathbf{R} \approx \mathbf{Q} \mathbf{A}_R \mathbf{Q}^T \) with \( \mathbf{A}_R = \text{diag} \{ \lambda_{max}, \ldots, \lambda_{min} \} \) being the diagonal matrix with the eigenvalues of \( \mathbf{R} \). Therefore, in this way \( \mathbf{R}_k \) and \( \mathbf{P}_k \) can be jointly diagonalized as \( \mathbf{P}_k = \mathbf{Q} \mathbf{A}_P \mathbf{Q}^T \) and \( \mathbf{R}_k \approx \mathbf{Q} \mathbf{A}_R \mathbf{Q}^T \), where \( \mathbf{Q} \) has the form [25]

\[
\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{Q} & -\mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{Q} & \mathbf{Q} \end{pmatrix}.
\]
The diagonal elements of this expression admit the recursion
\[ \gamma_{k+1} = \left[ I - 2E[\bar{\eta}]A_k + E[\eta^2] \left( \frac{2A_k}{A_k^T + r r^T} \right) \right] \gamma_k + \sum_k \frac{\eta_k}{\sigma_0^2}, \] (30)

The real-valued nature (decoupled real and imaginary parts) of the evolution of \( \kappa \) guarantees that the real and imaginary parts of \( \gamma_k \) evolve independently, so that we arrive at
\[ \Re \{ \gamma_k \} = A \Re \{ \gamma_k \} + \left| \gamma_k \right|^2 B \kappa_k - 2E[\bar{\eta}] \left| \eta_k \right|^2. \] (31)
Upon combining (25) and (31), we obtain the recursion for the augmented variable \( \tilde{\gamma}_k = \left[ \kappa_k, \Re \{ \gamma_k \} \right]^T \), which assumes the form
\[ \tilde{\gamma}_{k+1} = \left[ \begin{array}{cc} A & \B
\end{array} \right] \tilde{\gamma}_k + \left[ \begin{array}{cc}
\left| \eta_k \right|^2 B & A
\end{array} \right] \tilde{\gamma}_k. \] (32)

**Remark 6.** The recursion in (32) depends on both the standard and complementary convergence analyses, and reduces to the standard convergence of the ACLMS [25] for circular estimation error, that is, for \( \eta = 0 \).

### 5.4. Mean-square stability

For the recursion (32) to converge, the eigenvalues of \( A \) have to be less than unity. Instead of attempting to determine the eigenvalues of \( A \) directly, we use majorization inequalities of Hermitian block matrices [29], the Gantmacher theorem [30] and the Weyl inequality, to state that for a Hermitian positive semi-definite \( A \times 2 \) block matrix

\[ H = \begin{bmatrix}
M & K
N & \kappa
\end{bmatrix}, \] then

\[ \Re \{ \kappa \} = \Re \{ \kappa \} + \left| \kappa \right|^2 B \kappa_k - 2E[\bar{\eta}] \left| \eta_k \right|^2 \] (26)

and since \( \Re \{ \kappa \} \) is the condition that \( \Re \{ \kappa \} \) reduces to

\[ 1 - 2E[\bar{\eta}]r_{\min} + \left| \kappa \right|^2 E[\bar{\eta}] \left( 2r_{\max} + \Re \{ \kappa \} \right) < 1. \] (27)

The inequality still holds if \( \Re \{ \kappa \} \) is replaced by \( 2N r_{\max} \), and therefore the MICC-based stochastic gradient algorithm achieves mean-square stability for

\[ 0 < \mu < \frac{\pi \sigma^2 (1 - \left| \kappa \right|^2) r_{\min}}{(1 + \left| \kappa \right|^2) E[\kappa_{\sigma_{\eta}}(x_k)] (N + 1) r_{\max}^2}. \] (28)

Upon dividing the numerator and denominator with \( r_{\min} \) and recognizing that the maximum eigenvalue of the augmented covariance matrix, \( r_{\max} \), is the sum of the eigenvalues of the covariance, \( \Theta \), and pseudo-covariance, \( \Phi \), matrices that is, \( r_{\max} = \lambda_{\max} + \lambda_{\min} \), we have [25]

\[ 0 < \mu < \frac{\pi \sigma^2 (1 - \left| \kappa \right|^2)}{(1 + \left| \kappa \right|^2) E[\kappa_{\sigma_{\eta}}(x_k)] (N + 1) s \left\{ \kappa \right\} (\lambda_{\max} + \lambda_{\min})}, \] (29)

where the eigenvalue spread of the augmented covariance matrix is

\[ s \left\{ \kappa \right\} = \lambda_{\max} = \lambda_{\min} \] (30)

### 5.5. Steady-state analysis

The steady-state values of \( \kappa \) and \( \gamma \) are given by

\[ \kappa_{\infty} = \left[ I - A \right]^{-1} \left[ B \Re \{ \gamma^\omega_{\infty} \} + E[\bar{\eta}] \left( 1 + \left| \kappa \right|^2 \right) \sigma_0^2 \right], \] (31)

\[ \Re \{ \gamma^\omega_{\infty} \} = \left[ I - A \right]^{-1} \left[ \left| \gamma \right|^2 B \kappa_{\infty} - 2E[\bar{\eta}] \left| \eta \right|^2 \sigma_0^2 \right]. \] (32)

Upon combining (38) and (39), the steady-state misadjustment can be expressed as

\[ M_{\text{MCCC}} = \frac{r_{\infty} \kappa_{\infty}}{\sigma_0^2} = \left[ \left( I - A \right) - \left| \gamma \right|^2 B \left[ I - A \right]^{-1} B^T \right]^{-1} \times \] (33)

\[ E[\bar{\eta}] \left( 1 + \left| \kappa \right|^2 \right) \left[ I - 2B \left[ I - A \right]^{-1} \right]. \] (34)

**Remark 7.** The steady-state misadjustment of the MICC algorithm increases with the increase in noncircularity of the estimation error.

### 6. SIMULATIONS

The optimum weights in (7) were chosen arbitrarily as

\[ \mathbf{h}_{\text{opt}} = [1 - 2, -3 + 3 j]^T, \quad \mathbf{g}_{\text{opt}} = [2 + 0.5 j, -2 + 2 j]^T \] (35)

and the complex input signal, \( x_k \), was proper Gaussian noise. The real and imaginary parts of the noise, \( \eta_k \), were characterized by the respective pdfs \( 0.9 N(0, 1) \) and \( N(0, 10) \), where the large variance of 10 provides large impulsive disturbances.

The performance of the MCCC-based stochastic gradient (MICC) was compared to its “proper” MCCC-based counterpart (MCCC) [10] and the established CLMS and augmented CLMS (ACLASS) algorithms [31, 32, 33]. The weight signal-to-noise ratio (WSNR), defined as

\[ \text{WSNR}_{db} = 10 \log_{10} \left( \frac{\left\| \mathbf{w}_{\text{opt}} \right\|^2}{\left\| \mathbf{w}_{\text{opt}} - \mathbf{w} \right\|^2} \right) \] (36)
was used to quantify both convergence and misadjustment [8], where \( \mathbf{w}_{\text{opt}} = [h_k^T, g_k^T]^T \) is the weight vector computed at the time instant \( k \) for the widely linear algorithm (MICC and ACLASS), and \( \mathbf{w}_k = [h_k^T, 0]^T \) for the strictly linear algorithms (MCCC and CLMS).

Fig. 1 shows the average WSNR produced by 1000 Monte Carlo trials, with the initial value for the weights set to zero. The parameters in all algorithms considered were tuned such that their steady-state WSNR were equal in a Gaussian environment.

![Fig. 1: Weight signal-to-noise ratio (WSNR) of MICC, MCCC, CLMS and ACLASS under Gaussian proper noise (left panel) and impulsive improper noise (right panel).](image-url)
8. REFERENCES


