FIRST-ORDER PERTURBATION ANALYSIS OF SECSI WITH GENERALIZED UNFOLDINGS

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ABSTRACT

Tensor decompositions are regarded as a powerful tool for multi-dimensional signal processing. In this contribution, we focus on the well-known Canonical Polyadic (CP) decomposition and present a first-order perturbation analysis of the SEmi-algebraic framework for approximate CP decompositions via Smultaneous matrix diagonalization with Generalized Unfoldings (SECSI-GU), which is advantageous for tensors of an order higher than three. Numerical results indicate that the analytical relative Mean Square Factor Error (rMSFE) of the estimated factor matrices resulting from each generalized unfolding considered in SECSI-GU matches the empirical rMSFE very well. As SECSI-GU considers all possible partitions of the tensor modes resulting in a large number of candidate factor matrix estimates, an exhaustive search-based criterion to select the final factor matrix estimates leads to a prohibitive computational complexity. The accurate performance prediction achieved by the first-order perturbation analysis conducted in this paper will significantly facilitate the selection of the final factor matrix estimates in an efficient manner and will therefore contribute to a low-complexity enhancement of SECSI-GU.

Index Terms— Canonical Polyadic decomposition, generalized unfoldings, first-order perturbation analysis

1. INTRODUCTION

The $R$-way Canonical Polyadic (CP) decomposition, also known as Parallel Factor (PARAFAC) analysis \cite{Harshman} or Canonical Decomposition (CANDDECOMP) \cite{DeLeeuw}, has found applications in a variety of research fields, including array signal processing, wireless communications, and image processing \cite{Bader2007, Kolda2009, Nicola2010}. To accomplish the challenging task of computing an approximate CP decomposition of observed signals of interest in additive noise, a SEmi-algebraic framework for approximate CP decompositions based on Smultaneous matrix diagonalizations (SECSI) was proposed in \cite{Kolda2009, Bader2007, Aharon2006}. SECSI algebraically rephrases the CP decomposition into a set of less complex Simultaneous Matrix Diagonalization (SMD) problems. Combining generalized unfoldings with the idea of considering all possible generalized unfoldings to obtain multiple candidate CP models as in SECSI leads to SECSI with Generalized Unfoldings (SECSI-GU) \cite{Zitser2015}. For tensors with $R > 3$ dimensions, SECSI-GU enhances the identifiability and outperforms SECSI in terms of estimation accuracy, and it is very flexible in controlling the complexity-accuracy trade-off \cite{Zitser2015}. Such semi-algebraic approaches exhibit superiority over the Alternating Least Squares (ALS) procedure \cite{Kolda2009, DeLeeuw} which may require a large number of iterations and is sensitive to ill-conditioned data. In addition, the non-iterative nature of the semi-algebraic methods enables a parallelized implementation, which is not possible with iterative methods such as ALS.

Recently, the performance analysis of the truncated Higher-Order SVD (HOSVD) \cite{Loewen2015} and the approximate CP decomposition via SECSI \cite{Bader2007} for 3-D tensors have been carried out. These new advances in perturbation analysis of tensor decompositions further spark the interest in developing a performance analysis framework for SECSI-GU. Taking into account all possible generalized unfoldings, SECSI-GU accordingly forms and solves a large number of SMDs (up to $3^R - 3 \cdot 2^{R-1} + 3$ for an $R$-D tensor \cite{Zitser2015} and twice this amount if the considerations at the end of Section 3 are taken into account). Several heuristic selection criteria have been proposed to determine which SMDs to solve and how to select the final estimates of the factor matrices \cite{Zitser2015}. A performance analysis of SECSI-GU will enable us to predict the performance of SECSI-GU with respect to each of the possible generalized unfoldings and consequently to select the generalized unfolding leading to the “best” solutions in terms of, e.g., the minimum relative Mean Square Factor Error (rMSFE). Hence, conducting an analytical performance evaluation of SECSI-GU is not only of theoretical but also of practical interest.

In this paper, we present a first-order perturbation analysis of SECSI-GU. Note that SECSI-GU constructs the matrices for each SMD following the concept of the “Semi-Algebraic Tensor Decomposition” (SALT) algorithm \cite{Zitser2015}, later named DIAG (DIrect AlGo-rithm for canonical polyadic decomposition) \cite{Zitser2015A}, which is essentially different from that of SECSI. The performance analysis of SECSI-GU presented here has thus fundamental differences from that of SECSI in \cite{Cheng2015}. Moreover, the analytical performance evaluation of SECSI \cite{Cheng2015} was derived only for 3-D tensors, while the results shown in this contribution are applicable also to tensors with more than three dimensions. On the other hand, DIAG constructs only a single SMD from a single appropriately selected generalized unfolding, whereas all possible generalized unfoldings are considered in SECSI-GU, giving rise to multiple candidates of the factor matrix estimates. Owing to this fact, a performance evaluation of the DIAG algorithm is inherent in the proposed performance analysis framework for SECSI-GU. As the least-squares Khatri-Rao factorization \cite{Li2011} is employed in the final steps of SECSI-GU to obtain the estimates of the factor matrices \cite{Zitser2015}, we are able to utilize our previous results of its performance analysis \cite{Cheng2015B} to get the closed-form expression of the rMSFE. As we show via numerical simulations, it leads to a very accurate prediction for the rMSFE of SECSI-GU.

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Throughout this paper, the r-mode product between an R-way tensor with size $I_r$, along mode $r = 1, 2, \ldots, R$ represented as $A \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_R}$ and a matrix $U \in \mathbb{C}^{F_r \times I_r}$ is written as $A \times_r U$. It is computed by multiplying all r-mode vectors of $A$ with $U$, whereas the r-mode vectors of $A$ are obtained by varying the r-th index from 1 to $I_r$ and keeping all other indices fixed. Aligning all r-mode vectors as the columns of a matrix yields the r-mode unfolding of $A$ which is denoted by $[A]_{(r)} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_R}$. In other words, $[A \times_r U]_{(r)} = U \cdot [A]_{(r)}$. Here the reverse cyclical ordering of the columns, as proposed in [16], is used for the r-mode unfoldings. The tensor $I_{R,d}$ is an R-dimensional identity tensor of size $d \times d \times \ldots \times d$, which is equal to one if all $R$ indices are equal and zero otherwise. A $d \times d$ identity matrix, on the other hand, is denoted by $I_d$. Moreover, the Kronecker product between two matrices is expressed by $A \otimes B$ and the Khatri-Rao (column-wise Kronecker) product by $A \circ B$. The vectorization operation of a matrix is symbolized by $\text{vec}\{\cdot\}$. We use the superscript $^\dagger$ for the Moore-Penrose pseudo inverse of a matrix.

2. CP DECOMPOSITION VIA SECSI-GU

The CP decomposition of an R-way rank-$d$ tensor $X_0$ is written as

$$X_0 = I_{R,d} \times_1 F_1 \times_2 \ldots \times_R F_R \in \mathbb{C}^{M_1 \times M_2 \times \ldots \times M_R},$$

(1)

where $F_r \in \mathbb{C}^{M_r \times d}$ ($r = 1, 2, \ldots, R$) represent the factor matrices. Dividing the set of indices $(1, 2, \ldots, R)$ into a P-dimensional subset $\alpha^{(1)} = [\alpha_1, \alpha_2, \ldots, \alpha_P]$ and an $(R-P)$-dimensional subset $\alpha^{(2)} = [\alpha_{P+1}, \alpha_{P+2}, \ldots, \alpha_R]$ with $1 \leq P < R$, SECSI-GU considers generalized unfoldings

$$[X_0]_{(\alpha^{(1)},\alpha^{(2)})} = (F_{\alpha_1} \circ \cdots \circ F_{\alpha_P}) \cdot (F_{\alpha_{P+1}} \cdots \circ F_{\alpha_R})^T,$$

(2)

where the first $P$ indices are arranged into the rows and the remaining $R-P$ indices into the columns. Assigning the different modes into three non-empty groups yields

$$F_A = F_{\alpha_1} \circ \cdots \circ F_{\alpha_P} \in \mathbb{C}^{M_A \times d}$$

$$F_B = F_{\alpha_{P+1}} \circ \cdots \circ F_{\alpha_R} \in \mathbb{C}^{M_B \times d}$$

$$F_C = F_0 \in \mathbb{C}^{M_C \times d},$$

where $M_A = \prod_{t=1}^P M_{\alpha_t}$, $M_B = \prod_{t=1}^P M_{\alpha_{P-t+1}}$, and $M_C = \prod_{t=0}^{R-P} M_{\alpha_t}$, with $1 \leq t < P < R$. In addition, $\alpha_A$, $\alpha_B$, and $\alpha_C$ contain the indices assigned to each of the three groups, respectively, which appear later in the figures in Section 4. Consequently, the generalized unfolding in (2) can be written as

$$[X_0]_{(\alpha^{(1)},\alpha^{(2)})} = (F_A \circ F_B) \cdot F_C^T.$$  

(3)

For a certain generalized unfolding of the perturbed version of $X_0$ given by $X = X_0 + \tilde{N}$, we summarize SECSI-GU as follows:

- Compute the truncated SVD of $[X]_{(\alpha^{(1)},\alpha^{(2)})}$ in $\mathbb{C}^{M_A \times M_B \times M_C}$ and obtain

$$[X]_{(\alpha^{(1)},\alpha^{(2)})} \approx \tilde{U}[s] \cdot \tilde{\Sigma}[s] \cdot \tilde{V}[s]^H,$$

(4)

where $\tilde{U}[s] \in \mathbb{C}^{M_A \times M_B \times d}$, $\tilde{V}[s] \in \mathbb{C}^{M_C \times d}$, and $\tilde{\Sigma}[s] \in \mathbb{C}^{d \times d}$. The column space of $\tilde{U}[s]$ is an estimate of the column space of $[X]_{(\alpha^{(1)},\alpha^{(2)})}$. Also define $\tilde{\Sigma}[s] = \tilde{\Sigma}[s] \cdot \tilde{V}[s]^H \in \mathbb{C}^{d \times M_C}$.

- Partition $U[s] \in \mathbb{C}^{M_A \times M_B \times d}$ into $M_A$ blocks of size $M_B \times d$ denoted by $U_{m}[s]$, $m = 1, 2, \ldots, M_A$, such that

$$U[s] = \begin{bmatrix} U_1[s] & \cdots & U_{M_A}[s] \end{bmatrix}.$$  

(5)

- Construct a set of matrices $\tilde{\Gamma}_{p,m} = \tilde{U}_p[s] \cdot \tilde{U}_m^H \in \mathbb{C}^{d \times d}$, where $m = 1, 2, \ldots, M_A$, and $p$ is chosen according to

$$p = \arg \min_{n=1,2,\ldots,M_A} \text{cond} \left\{ \tilde{U}_n[s] \right\},$$

(6)

with cond $\{\cdot\}$ representing the condition number.

- Compute an (approximate) SMD of $\tilde{\Gamma}_{p,m} \in \mathbb{C}^{d \times d}$ ($m = 1, 2, \ldots, M_A$)

$$\tilde{\Gamma}_{p,m} \approx \hat{T} \cdot \hat{D}_m \cdot \hat{T}^{-1},$$

(7)

where $\hat{D}_m$ are diagonal matrices, thereby obtaining $\hat{T} \in \mathbb{C}^{d \times d}$, which approximates the two Khatri-Rao products $F_A \circ F_B$ and $F_C$ as follows:

$$\tilde{U}[s] \cdot \hat{T} \approx F_A \circ F_B = F_{\alpha_1} \cdots \circ F_{\alpha_P} \cdot \hat{F}_C = F_{\alpha_{P+1}} \cdots \circ F_{\alpha_R},$$

(8)

- Perform the least-squares Khatri-Rao factorization of $\tilde{U}[s] : \hat{T}$ and of $\tilde{Z}[s]^T : \hat{T}^{-T}$, respectively, to obtain estimates of the factor matrices $\hat{F}_r$ ($r = 1, 2, \ldots, R$).

3. PERFORMANCE ANALYSIS OF SECSI-GU

Let us denote the perturbations of $F_A \circ F_B$ and $F_C$ caused by additive noise as $\Delta (F_A \circ F_B)$ and $\Delta F_C$, respectively. In this section, we derive the closed-form expression of $\text{vec} \{\Delta (F_A \circ F_B)\}$ and $\text{vec} \{\Delta F_C\}$. They can be regarded as the input of the already established first-order perturbation analysis of the least-squares Khatri-Rao factorization [15] to finally obtain the closed-form expression of the rMSE of each factor matrix.

With $\Delta U[s]$ and $\Delta \hat{T}$ representing the perturbations of $U[s]$ and $\hat{T}$, respectively, as in

$$\tilde{U}[s] = U[s] + \Delta U[s] \quad \text{and} \quad \hat{T} = T + \Delta \hat{T},$$

(10)

$\Delta (F_A \circ F_B)$ can be expressed as

$$\Delta (F_A \circ F_B) = \Delta U[s] \cdot T + U[s] \cdot \Delta \hat{T} + \mathcal{O}(\Delta^2),$$

(11)

where $\mathcal{O}(\Delta^2)$ includes all terms with an order higher than one. The SVD of $[X_0]_{(\alpha^{(1)},\alpha^{(2)})}$ is given by

$$[X_0]_{(\alpha^{(1)},\alpha^{(2)})} = \begin{bmatrix} U[0] & \tilde{U}[s] \end{bmatrix} \cdot \begin{bmatrix} \Sigma[0] & 0 \\ 0 & \tilde{\Sigma}[s] \end{bmatrix} \cdot \begin{bmatrix} V[0]^T & V[s]^T \end{bmatrix},$$

where the columns of $U[0] \in \mathbb{C}^{M_A \times M_B \times d}$, $V[0] \in \mathbb{C}^{M_C \times d}$, and $U[s] \in \mathbb{C}^{M_A \times (M_B \times d)}$ span the column space, row space, and null space of $[X_0]_{(\alpha^{(1)},\alpha^{(2)})}$, respectively. In addition, $\Sigma[s] \in \mathbb{C}^{d \times d}$ is a diagonal matrix whose diagonal elements are the $d$ non-zero singular values of $[X_0]_{(\alpha^{(1)},\alpha^{(2)})}$. Based on the first-order perturbation analysis of the SVD [17], we have

$$\Delta U[s] = \mathcal{T}[s] \cdot [\tilde{N}]_{(\alpha^{(1)},\alpha^{(2)})} \cdot V[s] \cdot \Sigma[s]^{-1} + \mathcal{O}(\Delta^2),$$

(12)
where \( \Upsilon[n] = U[n] \cdot U[n]^H \in \mathbb{C}^{M_A \times M_B} \) is the projection matrix into the noise subspace of \([X_0]\)\((\alpha(1), \alpha(2))\), and \([N]\)\((\alpha(1), \alpha(2))\) is the generalized unfolding of \(N\). The vectorization of \(\Delta U[n]\) is obtained as
\[
\text{vec} \left\{ \Delta U[n] \right\} = \left( \Sigma \right)^{-1} \cdot V[n]^T \otimes Y[n] \cdot \text{vec} \left\{ \left[ N \right]_{(\alpha(1), \alpha(2))} \right\} + O(\Delta^2).
\]
Assume that the covariance matrix \(R_{mn}\) of the vectorization of the 1-mode unfolding of the noise tensor \(N\), given by \(n_1 = \text{vec} \left\{ \left[ N \right]_{(1)} \right\}\), is known. We now define a permutation matrix \(P \in \{0, 1\}^{M \times M}\) that satisfies \(\text{vec} \left\{ \left[ N \right]_{(\alpha(1), \alpha(2))} \right\} = P \cdot n_1\) and consequently write the vectorization of \(\Delta U[n]\) in the form of
\[
\text{vec} \left\{ \Delta U[n] \right\} = K_0 \cdot n_1 + O(\Delta^2),
\]
where \(K_0 = \left( \Sigma \right)^{-1} \cdot V[n]^T \otimes Y[n] \cdot P \in \mathbb{C}^{d \cdot M_A \times M_B}\).

To compute \(T\) in SECSI-GU, we employ the JDTM (Joint Diagonalization based on Targeting Hyperbolic Matrices) algorithm [18] for the SMD of \(\Gamma_{p,m}\) in \(\mathbb{C}^{d \times d}\) \((m = 1, 2, \ldots, M_A)\). According to the performance analysis of JDTM in [19], we have
\[
\text{vec} \left\{ \Delta T \right\} = -A^* \cdot B \cdot \gamma + O(\Delta^2),
\]
where
\[
A = \begin{bmatrix} A_1 & \cdots & A_A \end{bmatrix}, \quad B = I_{M_A} \otimes B_0, \quad \gamma = \begin{bmatrix} \gamma_1 \cdots \gamma_{M_A} \end{bmatrix}.
\]

Here a selection matrix is defined as \(J(d) \in \{0, 1\}^{d^2 \times d^2}\) such that \(\text{vec} \left\{ \text{Off}(X) \right\} = J(d) \cdot \text{vec} \left\{ X \right\}\), where the \(\text{Off}()\) operator sets the diagonal elements of its input matrix to zeros. Note that \(\Gamma_{p,m}\) and \(\Delta_{p,m}\) are constructed from the noiseless tensor \(X_0\) and can be regarded as the “true” version of \(\Gamma_{p,m}\) and \(\Delta_{p,m}\), respectively. To derive the perturbation of \(\Gamma_{p,m}\) denoted by \(\Delta \Gamma_{p,m}\), in (16), let us write \(\Gamma_{p,m}\) in the following form
\[
\Gamma_{p,m} + \Delta \Gamma_{p,m} = \left( U_p^{[1]} + \Delta U_p^{[1]} \right)^T \otimes \left( U_m^{[1]} + \Delta U_m^{[1]} \right) - \left( \Sigma \right)^{-1} \cdot V[n]^T \cdot \left( \Sigma \right)^{-1} \cdot V[n]^T \otimes Y[n].
\]

Based on [20], the matrix pseudo inversion in (17) is expressed as
\[
\left( U_p^{[1]} + \Delta U_p^{[1]} \right)^T = U_p^{[1]^T} + \Delta U_p^{[1]^T} + O(\Delta^2).
\]
Inserting (18) into (17), we obtain the perturbation \(\Delta \Gamma_{p,m}\) as
\[
\Delta \Gamma_{p,m} = U_p^{[1]^T} \cdot \Delta U_p^{[1]} - \Delta U_p^{[1]^T} \cdot U_p^{[1]} - \Delta U_p^{[1]^T} \cdot U_p^{[1]^T} + O(\Delta^2).
\]
Define two block selection matrices \(J_{n,m}\) and \(J_p\) such that \(\Delta U_n^{[1]} = J_n \cdot \Delta U_p^{[1]}\) and \(\Delta U_p^{[1]} = J_p \cdot \Delta U_p^{[1]}\), respectively. Consequently, \(\gamma_m = \text{vec} \left\{ \Delta \Gamma_{p,m} \right\}\), as given in (16), is now written as
\[
\gamma_m = \left( I_d \otimes \left( U_p^{[1]^T} \cdot J_p \right) \right) \cdot \text{vec} \left\{ \Delta U_p^{[1]} \right\} + O(\Delta^2).
\]
Substituting \(\text{vec} \left\{ \Delta U_p^{[1]} \right\}\) in (20) by (14) allows us to express \(\gamma\), where \(\gamma_m (m = 1, 2, \ldots, M_A)\) are stacked, as
\[
\gamma = K_2 \cdot n_1 + O(\Delta^2),
\]
with \(K_2 = \begin{bmatrix} K_1^{(1)^T} & \cdots & K_1^{(M_A)^T} \end{bmatrix}^T\) containing the stacking of \(K_1^{(m)} \in \mathbb{C}^{d^2 \times M} (m = 1, 2, \ldots, M_A)\) defined via
\[
K_1^{(m)} = \left( I_d \otimes \left( U_p^{[1]^T} \cdot J_m \right) \right) - \left( U_p^{[1]^T} \cdot U_p^{[1]^T} \right) \otimes \left( U_p^{[1]^T} \cdot J_p \right) \cdot K_0.
\]
Accordingly, we obtain \(\text{vec} \left\{ \Delta T \right\}\) first given in (15) as
\[
\text{vec} \left\{ \Delta T \right\} = K_3 \cdot n_1 + O(\Delta^2),
\]
where \(K_3 = -A^* \cdot B \cdot K_2 \in \mathbb{C}^{d^2 \times M}\).

The vectorization of \(\Delta (F_A \odot F_B)\) takes the form
\[
\text{vec} \left\{ \Delta (F_A \odot F_B) \right\} = \left( T \otimes I_{M_A \times M_B} \right) \cdot \text{vec} \left\{ \Delta U_p^{[1]} \right\} + \left( I_d \otimes U_p^{[1]} \right) \cdot \text{vec} \left\{ \Delta T \right\} + O(\Delta^2).
\]
By inserting (14) and (22) into (23), we have
\[
\text{vec} \left\{ \Delta (F_A \odot F_B) \right\} = K_4 \cdot n_1 + O(\Delta^2),
\]
where \(K_4 = \left( T \otimes I_{M_A \times M_B} \right) \cdot K_0 + \left( I_d \otimes U_p^{[1]} \right) \cdot K_3\).

In the following, we proceed to derive \(\text{vec} \left\{ \Delta F_C \right\}\). To this end, let us express \(F_C\) as
\[
F_C + \Delta F_C = \left( Z^{[1]} + \Delta Z^{[1]} \right) \cdot (T + \Delta T)^{-T},
\]
where \(\Delta Z^{[1]}\) denotes the perturbation of \(Z^{[1]}\) such that \(Z^{[1]} = Z^{[1]} + \Delta Z^{[1]}\). Owing to the fact that [20]
\[
(T + \Delta T)^{-1} = T^{-1} - \Delta T \cdot T^{-1} + O(\Delta^2),
\]
\(\Delta F_C\) can be further written as
\[
\Delta F_C = \Delta Z^{[1]} \cdot T^{-1} - Z^{[1]} \cdot T^{-1} \cdot \Delta T \cdot T^{-1} + O(\Delta^2).
\]
Vectorizing \(\Delta F_C\) leads to
\[
\text{vec} \left\{ \Delta F_C \right\} = \left( T^{-1} \otimes I_{M_C} \right) \cdot \text{vec} \left\{ \Delta Z^{[1]} \right\} + O(\Delta^2)
\]
\[
- \left( T^{-1} \otimes \left( Z^{[1]} \cdot T^{-1} \right) \right) \cdot \text{vec} \left\{ \Delta T \right\}.
\]
Defining a permutation matrix \(P_{d,d} \in \{0, 1\}^{d^2 \times d^2}\) satisfying \(\text{vec} \left\{ \Delta T \right\} = P_{d,d} \cdot \text{vec} \left\{ \Delta T \right\}\), allows us to use vec \(\Delta T\) already obtained in (22). To derive vec \(\Delta Z^{[1]}\) for the final expression of vec \(\Delta F_C\) given in (27), we rewrite \(Z^{[1]}\) originally defined as \(Z^{[1]} = \Sigma^{[1]} \cdot V[n]^H\) into
\[
\hat{Z}^{[1]} = Z^{[1]} + \Delta Z^{[1]} = \hat{U}_B^{[1]^T} \cdot [X]_{(\alpha(1), \alpha(2))}.
\]
The resulting expression of \(\Delta Z^{[1]}\)
\[
\Delta Z^{[1]} = [X]_{(\alpha(1), \alpha(2))} \cdot \Delta U_B^{[1]} + [N]_{(\alpha(1), \alpha(2))} \cdot U_B^{[1]} + O(\Delta^2).
\]
is further simplified into
\[
\Delta Z^{[s]}_T = [\mathbb{N}]_T^{(1), (2)} \cdot U^{[s]^*} + O(\Delta^2)
\]
due to the observation that the first term \([\mathbb{X}]_0^{T/(1), (2)} \cdot \Delta U^{[s]^*}\) is a zero matrix according to the definition of \(\Delta U^{[s]}\) in (12). Taking the vectorization of \(\Delta Z^{[s]^*}\) gives
\[
\text{vec}\{\Delta Z^{[s]^*}\} = K_5 \cdot n_1 + O(\Delta^2),
\]
where
\[
K_5 = (U^{[s]} \otimes I_{MC}) \cdot P \in \mathbb{C}^{dMC \times M} \quad \text{and} \quad P \in \{1, 0\}^{M \times M} \text{ is a permutation matrix leading to vec}\{[\mathbb{N}]_T^{(1), (2)}\}
\] = \(P \cdot n_1\).

Finally, vec\{\Delta F_C\} takes the form
\[
\text{vec}\{\Delta F_C\} = K_6 \cdot n_1 + O(\Delta^2),
\]
where
\[
K_6 = (T^{-1} \otimes I_{MC}) \cdot K_5 - (T^{-1} \otimes \left(Z^{[s]^*} \cdot T^{-T}\right)) \cdot P_{(d,d)} \cdot K_3 \in \mathbb{C}^{dMC \times M}.
\]

Taking vec\{\Delta (F_A \circ F_B)\} and vec\{\Delta F_C\} as the input of the performance analysis of the least-squares Khatri-Rao factorization [15], we are able to obtain the first-order perturbation of \(F_C\) denoted by \(\Delta F_C\). Its vectorization vec\{\Delta F_C\} can be expressed as vec\{\Delta F_C\} = \(K \cdot n_1 + O(\Delta^2)\) similar to the vectorization of the perturbation terms derived above. Subsequently, we get the closed-form expression of the rMSFE for each factor matrix in terms of the second-order moments of the noise, i.e., the covariance matrix \(R_{nn}\) with respect to \(n_1\). For detailed derivations and the resulting explicit expression of \(K\) as well as the rMSFE, the reader is referred to [15].

It is worth noting that \(T\) can be alternatively obtained via the joint diagonalization of another set matrices given by \(\Omega_{p, n_1} = Z^{[s]}_m \cdot Z^{[s]}_m^* \in \mathbb{C}^{d \times d}\), where \(Z^{[s]}_m \in \mathbb{C}^{d \times (M^{(1)} \cdot M^{(2)})}\) \((m = 1, 2, \ldots, M^{(1)})\) are obtained by partitioning \(Z^{[s]}\) as
\[
Z^{[s]} = \left[ Z_1^{[s]} \ldots Z_{M^{(1)} m}^{[s]} \right]
\]
with \(M^{(1)} = \prod_{r=p+1}^{R} M^{(r)}\) and \(M^{(2)} = \prod_{q=p+1}^{R} M^{(q)}\), i.e., \(M_C = M^{(1)} \cdot M^{(2)}\). The index \(p\) corresponds to the block \(Z^{[s]}_p\) that has the minimum condition number. Due to space limitations, the first-order perturbation analysis of SECSI-GU where the SMDs are constructed as mentioned above is not included in this paper.

4. SIMULATION RESULTS
To demonstrate the validity of our first-order performance analysis of SECSI-GU, we present comparisons between the analytical total rMSFE and empirical ones obtained via Monte Carlo simulations. The factor matrices \(F_r\) \((r = 1, 2, \ldots, R)\) contain elements drawn independently from a zero-mean Gaussian distribution with unit variance, while elements of the noise tensor \(\mathbb{N}\) were drawn similarly with variance \(\sigma_n^2\). Accordingly, we define SNR = 1/\(\sigma_n^2\). Fig. 1 depicts the results for a real-valued case where \(R = 4, M_1 = 4, M_2 = 7, M_3 = 6, M_4 = 4,\) and \(d = 4\). A complex-valued case is considered in Fig. 2, where \(R = 5, M_r = 4 (r = 1, \ldots, R),\) and \(d = 4\). In the \(R = 4\) and \(R = 5\) scenarios, the total number of generalized unfoldings to be considered reaches 36 and 150, respectively. For clarity of the figures, only the results with respect to a few generalized unfoldings are shown as representative examples. In both cases, a good match between the analytical and empirical results is evident, especially in the higher SNR regime.

5. CONCLUSION
We have presented a first-order perturbation analysis of SECSI-GU for the approximate CP decomposition of noise-corrupted tensors of an order higher than three. To obtain the closed-form expression of the rMSFE for each factor matrix, we derived the first-order perturbations of all intermediate outcomes at every step of the SECSI-GU framework, ranging from the formulation of target matrices for the SMDs, to the estimation of the Khatri-Rao products of the factor matrices. Simulation results show that our performance analysis of SECSI-GU is able to predict the rMSFEs for each possible generalized unfolding very accurately, especially in the higher SNR regime. For future work, an efficient selection scheme for SECSI-GU will be designed, which determines the generalized unfolding to be considered based on the performance prediction provided by this first-order perturbation analysis. It will avoid computing all possible factor matrix estimates corresponding to all possible generalized unfoldings.
6. REFERENCES


