ON THE $L^4$ CONVERGENCE OF PARTICLE FILTERS WITH GENERAL IMPORTANCE DISTRIBUTIONS

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ABSTRACT
In this paper we extend the $L^4$ proof of Hu et al. (2008) from bootstrap type of particle filters to particle filters with general importance distributions. The result essentially shows that with general importance distributions the particle filter converges provided that the importance weights are bounded. By numerical simulations we also show that this condition is often also a practical requirement for a good performance of a particle filter.

Index Terms— Particle filter; convergence; importance distribution; unbounded function

1. INTRODUCTION
Particle filters [1, 2, 3] are powerful methods for approximate Bayesian filtering in state space models of the form

$$x_t \sim f(x_t \mid x_{t-1}), \quad y_t \sim g(y_t \mid x_t), \quad t = 1, \ldots, N,$$

where $x_t \in \mathbb{R}^n$ is the state of the system, $y_t \in \mathbb{R}^m$ is the measurement, $f(x_t \mid x_{t-1})$ is the transition probability density (w.r.t. Lebesgue measure) modeling the dynamics of the system, and $g(y_t \mid x_t)$ is the conditional probability density of measurements modeling the distribution of measurements.

Particle filters form a weighted set of Monte Carlo samples $\{(x^i_t, w^i_t) : i = 1, \ldots, N\}$ such that the posterior expectation of a test function $\phi(\cdot)$ can be approximated as

$$E[\phi(x_t) \mid y_{1:t}] \approx \sum_{i=1}^N w^i_t \phi(x^i_t). \quad (2)$$

A particle filter converges if, in a suitable sense, the above approximation becomes exact when $N \to \infty$.

Various types of convergence result for particle filters with general importance distributions, but with bounded test functions can be found in the survey article [4]. Long-term stability results and central limit theorem type of convergence theorems for particle filters (also for unbounded functions), can be found in [5, 6, 7, 8] and references therein. $L^4$ type of convergence results for the unbounded case have recently been studied in [9, 10], but only in the case of bootstrap type of importance distributions.

In this paper, we extend the proof of Hu et al. (2008) [9] to the case of general importance distributions. The results show that the boundedness of importance weights (along with the model densities) is a sufficient condition for the convergence also in the case of unbounded test functions, which is also a sufficient condition in the bounded case [4]. We also discuss the practical implications of the condition to certain importance distributions proposed in literature.

2. PARTICLE FILTERING
Recall that the Bayesian filter for the state space model in (1) can be written in the abstract form [9]:

$$\pi_t \mid x_{t-1}, \phi = (\pi_{t-1} \mid x_{t-1}, f \phi), \quad (3)$$

$$\pi_t \mid x_t, \phi = (\pi_{t-1} \mid x_{t-1}, \phi g) \left(\pi_{t-1} \mid x_{t-1}, g\right), \quad (4)$$

where we have defined

$$(\pi, \phi) = \int \phi(x) \pi(dx), \quad f = \int f(x \mid x_{t-1}) \phi(x) dx_t.$$  

With this notation, we obtain the bootstrap filter simply by replacing the measures $\pi$ with their finite-sample approximations and by introducing an additional resampling step. This was the starting point of the analysis in [9].

However, here we wish to analyze the convergence of the more general particle filter which does not correspond to a direct finite-sample approximation of the prediction and update steps above. Instead of sampling from the dynamic model distribution we sample from an importance distribution $q(x_t \mid x_{t-1}, y_t)$ and then compute weights for the samples. For this purpose it is convenient to rewrite the Bayesian filter as a single step

$$\pi_t \mid x_t, \phi = \frac{(\pi_{t-1} \mid x_{t-1}, \rho q), \phi)}{(\pi_{t-1} \mid x_{t-1}, \rho q), 1}, \quad (5)$$

where we have defined the importance weights

$$\rho(x_t, x_{t-1}) = \frac{g(y_t \mid x_t) f(x_t \mid x_{t-1})}{q(x_t \mid x_{t-1}, y_t)}.$$  

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As in [9, 10], to be able to cope with unbounded functions, we need to use a slightly modified version of the standard particle filter in order to guarantee the convergence. The modified particle filter is constructed such that we always have $((\pi_{t-1}^N, \rho q), 1) \geq \gamma_t > 0$, where $\gamma_t > 0$ is a chosen threshold [9]. The modified algorithm is the following.

**Algorithm 2.1** (General Modified Particle Filter). The algorithm is similar to [9], but includes importance distributions.

1. Initialize the particles, $x_0^i \sim \pi_0(dx_0)$

2. Draw samples according to $x_t^i \sim \sum_{j=1}^N \alpha_t^i \pi_{t|t-1}(x_t^i \mid x_{t-1}^j, y_t)$, where $\alpha_t^i$ are non-negative weights such that $\sum_{i=1}^N \alpha_t^i = 1$, $\sum_{i=1}^N \rho_t^i = 1$, and $\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \alpha_t^i q(x_t^i \mid x_{t-1}^j, y_t) = \frac{1}{N} \sum_{j=1}^N q(x_t^i \mid x_{t-1}^j, y_t)$.

3. If $((\pi_{t-1}^N, \rho q), 1) \geq \gamma_t$, proceed to step 4 otherwise return to step 2. Note that $\tilde{\rho}$ and $\tilde{q}$ are the values evaluated at $\tilde{x}_t^i$.

4. Rename $\tilde{x}_t^i = \tilde{x}_t^i$, and compute and normalize the weights $w_t^i = \rho(\tilde{x}_t^i, x_{t-1}^i), \tilde{w}_t^i = w_t^i / \sum_{i=1}^N w_t^i$.

5. Resample, $x_t^i \sim \tilde{\pi}_{t|t}^N(dx_t) = \sum_{i=1}^N \tilde{w}_t^i \delta_{\tilde{x}_t^i}(dx_t)$.

6. Set $t = t + 1$ and repeat from step 2.

### 3. CONVERGENCE WITH GENERAL IMPORTANCE DISTRIBUTION

To prove the convergence of the particle filter, we need to impose the following conditions (cf. [9]).

- **H0:** For any given $y_{1,t}$, we have $((\pi_s \mid x_{s-1}, \rho q), 1) > 0$, where $s = 1, \ldots, t$.

- **H1:** The dynamic model $f$, measurement model $g$, and the importance weights $\rho_t(x_t^i, x_{t-1}^i)$ are bounded. That is, there exist constants $C_f, C_g$, and $C_\rho$ such that $\|f\| \leq C_f$, $\|g\| \leq C_g$, and $\|\rho\| \leq C_\rho$, where the first norm is an operator norm induced by the supremum norm, and the second two are supremum norms of the functions.

- **H2:** The function $\phi(\cdot)$ satisfies $\sup_{x_s} |\phi(x_s)| \leq C(y_{1,t})$.

The main convergence theorem is the following.

**Theorem 3.1.** Consider the general modified particle filter algorithm and suppose that the conditions H0, H1, and H2 above hold. Then we have the following.

1. For a sufficiently large $N$, the algorithm will not run into an infinite loop on steps 2-3.

2. Let $L_1^t(g)$ be the class of functions satisfying H2. For any $\phi \in L_1^t(g)$, there exists a constant $C_t(\gamma_t, \rho)$, independent of $N$ such that

   $$\mathbb{E} \left[ \left( \left( \pi_{t|t}^N, \phi \right) - \left( \pi_{t|t}^N, \phi \right) \right)^4 \right] \leq C_t(\gamma_t, \rho) \frac{\|\phi\|_{L_1^t}^4}{N^2},$$

   where $\|\phi\|_{L_1}^4$ is defined as [9] $\|\phi\|_{L_1}^4 = max \{ 1, (\pi \| \phi \|^{1/4}, s = 0, 1, \ldots, t \}$. (8)

   **Proof.** The proofs for initialization and resampling steps are the same as in [9]. Thus, here, we only prove the convergence of the (combined) prediction and update steps. That is, we prove the convergence of the following:

   $$\left( \pi_{t|t}^N, \phi \right) - \left( \pi_{t|t}^N, \phi \right) = \left( \hat{\pi}_{t|t}^N, \phi \right) - \left( \hat{\pi}_{t|t}^N, \phi \right),$$

   where $\hat{\pi}_{t|t}^N = (\pi_{t-1|t-1}, \rho q^N)$ and $\hat{\pi}_{t|t}^N = (\pi_{t-1|t-1}, \rho q)$. It is now enough to study the bounded for the following terms:

   $$\mathbb{E} \left[ \left( \left( \pi_{t|t}^N, \phi \right) - \left( \pi_{t|t}^N, \phi \right) \right)^4 \right] \text{ and } \mathbb{E} \left[ \left( \left( \pi_{t|t}^N, |\phi|^4 \right) \right) \right].$$

   At $t = 0$, we have the initialization step for which the proof can be found in [9]. Next, we assume that there exist constants $C_{t-1|t-1}$ and $M_{t-1|t-1}$ such that the following is true:

   $$\sum_{t=0}^{t-1} \left( \pi_{t|t-1}^N, \phi \right) \leq \frac{C_t(\gamma_t, \rho)}{N} \left( \pi_{t|t-1}^N, |\phi|^4 \right), \quad (11)$$

   and

   $$\sum_{t=0}^{t-1} \left( \pi_{t|t-1}^N, |\phi|^4 \right) \leq M_{t-1|t-1} \left( \pi_{t|t-1}^N, |\phi|^4 \right). \quad (12)$$

   To study the boundedness of (10), we start by studying the numerator terms in (9). We first derive the bound for $\mathbb{E}[(\hat{\pi}_{t|t}^N, \phi) - (\hat{\pi}_{t|t}^N, \phi)]$ and then for $\mathbb{E}[(\hat{\pi}_{t|t}^N, |\phi|^4)]$.

   Let $\mathcal{F}_{t-1}$ be the $\sigma$-algebra generated by $x_{t-1}^i$. Then we can write $(\hat{\pi}_{t|t}^N, \phi) - (\hat{\pi}_{t|t}^N, \phi) = \Pi_1 + \Pi_2 + \Pi_3$, where

   $$\Pi_1 = (\hat{\pi}_{t|t}^N, \phi) - \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[\phi(\tilde{x}_t^i) \rho(\tilde{x}_t^i, x_{t-1}^i) \mid \mathcal{F}_{t-1}],$$

   $$\Pi_2 = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[\phi(\tilde{x}_t^i) \rho(\tilde{x}_t^i, x_{t-1}^i) \mid \mathcal{F}_{t-1}] - \frac{1}{N} \sum_{i=1}^{N} \pi_{t-1|t-1}^N \phi f g,$$

   $$\Pi_3 = \frac{1}{N} \sum_{i=1}^{N} \pi_{t-1|t-1}^N \phi f g - (\hat{\pi}_{t|t}^N, \phi).$$

   (13)
We treat the terms of $\Pi_1$, $\Pi_2$ and $\Pi_3$ separately and, in each case, we assume that $\|p\| \leq C_\rho$, and that $\|f\|$ and $\|g\|$ are bounded by some constants. Let $\tilde{x}_i \sim (\pi_{t-1}^{N, \alpha_i}, g)$, then
\[
E[\phi(\tilde{x}_i) \rho(\tilde{x}_i, x_{t-1}) \mid F_{t-1}] = (\pi_{t-1}^{N, \alpha_i}, f \phi g),
\]
\[
= \frac{1}{N} \sum_{i=1}^{N} (\pi_{t-1}^{N, \alpha_i}, f \phi g).
\]
(16)

The probability of the threshold $\gamma_t$ corresponds to event $A_t$ defined as $A_t = \{ (\pi_{t-1}^{N}, f \phi g) \geq \gamma_t \}$, where, by (16), we have
\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} \rho(\tilde{x}_i, x_{t-1}) \mid F_{t-1} \right] = (\pi_{t-1}^{N, \alpha_i}, f \phi g).
\]
(17)

Suppose $\|f\|$ and $\|g\|$ are bounded, and $H_0$ holds. Then, using Markov's inequality and (11), it implies that
\[
P \left[ \frac{1}{N} \sum_{i=1}^{N} \rho(\tilde{x}_i, x_{t-1}) \leq \gamma_t \mid F_{t-1} \right] \leq \frac{C_{t-1} \|f\|^{4} \|g\|^{4}}{N^2} \gamma_t - (\pi_{t-1}^{N, \alpha_i}, f \phi g)
\]
\[
= \tilde{C}_{t} \frac{\gamma_t}{N^2} = \epsilon. \quad (18)
\]

To bound (13), we use Lemmas 7.1, 7.2, 7.3, and 7.5 from [9], (16), and (12), which leads to
\[
E[\Pi_1^4 \mid F_{t-1}]
\]
\[
\leq \frac{2^4}{N^2} \left[ \sum_{i=1}^{N} E \left[ \phi(\tilde{x}_i) \rho(\tilde{x}_i, x_{t-1}) \mid F_{t-1} \right] \right]^{2}
\]
\[
\leq \frac{2^4}{(1 - \epsilon)^2} \left[ \frac{C^3_\rho (\pi_{t-1}^{N, \alpha_i}, f \phi g)}{N^2} \right]
\]
\[
+ \frac{2^4}{N^2} \left[ \sum_{i=1}^{N} \frac{\phi(\tilde{x}_i) \rho(\tilde{x}_i, x_{t-1})}{1 - \epsilon} \mid F_{t-1} \right]^{2}
\]
\[
\leq \frac{2^5 \tilde{C}_t}{(1 - \epsilon)^2} \left[ M_{t-1} \|\phi\|_{t-1,4}^4 \right] = \tilde{C}_{t} \frac{\|\phi\|_{t-1,4}^4}{N^2}. \quad (19)
\]

For the bound of (14), we use Lemmas 7.3 and 7.5, from [9], Eqs. (12), (16), and (18), as well as Jensen's inequality, which leads to
\[
E \left[ \frac{\Pi_2^4}{\|F_{t-1}\|^4} \right]
\]
\[
\leq \frac{2^4 e^2 C_\rho^2}{(1 - \epsilon)^4 N^2} \frac{E \left[ \left( (\pi_{t-1}^{N}, f \phi g) \right)^2 \right]}{\|F_{t-1}\|^2}
\]
\[
\leq \frac{2^4 e^2}{(1 - \epsilon)^4 N^2} \tilde{C}_t \frac{\|f\|^2 \|g\|^2 \|\phi\|_{t-1,4}^4}{N^2}.
\]
(20)

For the bound of (15), we use (11), which gives:
\[
E \left[ \frac{\Pi_3^4}{\|F_{t-1}\|^4} \right]
\]
\[
\leq \|f\|^4 \|g\|^4 E \left[ \left| (\pi_{t-1}^{N}, f \phi g) - (\pi_{t-1}^{N}, f \phi g) \right|^4 \right]
\]
\[
\leq \tilde{C}_{t} \frac{\|f\|^4 \|g\|^4 \|\phi\|_{t-1,4}^4}{N^2} = \tilde{C}_{t} \frac{\|\phi\|_{t-1,4}^4}{N^2}. \quad (21)
\]

By combining Eqs. (19), (20), and (21) via Minkowski's inequality, we get
\[
E \left[ \left| (\hat{\pi}_{t|t}, f \phi g) - (\hat{\pi}_{t|t}, f \phi g) \right|^4 \right] \leq \tilde{C}_{t} \frac{\|\phi\|_{t-1,4}^4}{N^2}. \quad (22)
\]

The bound for $E[(\hat{\pi}_{t|t}, f \phi g)]$ can be derived using the same technique as above, which leads to
\[
E \left[ \left| (\hat{\pi}_{t|t}, f \phi g) \right|^4 \right] \leq M_{t|t} \|\phi\|_{t-1,4}^4. \quad (23)
\]

Note that if we set $\phi = 1$ in (22) and (23), we get similar bounds for the difference of the denominators.

We finally study the boundedness of (10). For $E[(\pi_{t|t}, f \phi g) - (\pi_{t|t}, f \phi g)^4]$, we use (22) and (23) with $\phi = 1$ along with Minkowski's inequality, to get
\[
E \left[ \left| (\pi_{t|t}, f \phi g) - (\pi_{t|t}, f \phi g)^4 \right|^4 \right] \leq \frac{(\pi_{t|t}, f \phi g)}{\gamma_t (\pi_{t|t}, 1)} \left( \frac{1}{N^2} \right)^\frac{1}{2}
\]
\[
+ \frac{\tilde{C}_{t}^4}{(\pi_{t|t}, 1)} \left( \frac{\|\phi\|^2_{t-1,4}}{N^2} \right)^\frac{1}{2} \leq \frac{1}{N^1/2} \tilde{C}_{t} \frac{\|\phi\|_{t-1,4}}{N^2},
\]
which thus gives
\[
E \left[ \left| (\pi_{t|t}, f \phi g) - (\pi_{t|t}, f \phi g)^4 \right|^4 \right] \leq \tilde{C}_{t} \frac{\|\phi\|^4_{t-1,4}}{N^2}. \quad (24)
\]
For $\mathbb{E}(\pi_{t|t}^N | \phi^4)$, we similarly get

$$
\mathbb{E}\left[(\pi_{t|t}^N | \phi^4) - (\pi_{t|t}^c | \phi^4)\right] \leq \tilde{M}_{t|t} \phi^4_{t-1,4},
$$

which thus completes the proof.

\section{4. PRACTICAL IMPLICATIONS}

Our result states that provided that $||f||, ||g||, \text{ and } ||\rho||$ are bounded, we can ensure the convergence. The boundedness of $f$ and $g$ is indeed quite natural, but let’s take a closer look at the boundedness of the term $\rho$, which we defined in (6), on some commonly used importance distributions.

- The \textit{optimal importance distribution} [2] $q(x_t | x_{t-1}, y_t) = p(x_t | x_{t-1}, y_t)$ leads to $\rho(x_t, x_{t-1}) = \int g(y_t | x_t) f(x_t | x_{t-1}) \mathrm{d}x_t$ which is guaranteed to be bounded provided that $f$ and $g$ are bounded.

- In the \textit{bootstrap filter} [1, 9] we select $q(x_t | x_{t-1}, y_t) = f(x_t | x_{t-1})$, which gives $\rho(x_t | x_{t-1}) = g(y_t | x_t)$ and thus is ensured to be bounded.

- Using \textit{non-linear Kalman filters} to approximate the optimal importance distribution [2, 11, 12] gives $q(x_t | x_{t-1}, y_t) = \mathcal{N}(x_t | m_t, P_t)$ were, $m_t$ and $P_t$ are mean and covariance computed by the Kalman filter. We can now assure convergence only if the ratio of the optimal importance distribution and its approximation $p(x_t | x_{t-1}, y_t)/\mathcal{N}(x_t | m_t, P_t)$ is bounded. This requires that the tails of $p(x_t | x_{t-1}, y_t)$ are not heavier than the tails of the Gaussian distribution and that the covariance $P_t$ is bounded from below.

- We can also use a multivariate Student’s $t$-distribution with the parameters $m_t$ and $P_t$ above instead of the Gaussian distribution [13]. If we choose the degree of freedom in the Student’s $t$-distribution to be low enough, then $\rho$ can be assured to be bounded.

\textbf{Example 4.1} (Linear Gaussian state space model). Consider the one-dimensional Gaussian random walk model

$$
x_t = x_{t-1} + q_{t-1}, \quad q_{t-1} \sim \mathcal{N}(0, Q), \quad y_t = x_t + r_t, \quad r_t \sim \mathcal{N}(0, R).
$$

The \textit{optimal importance distribution} is now Gaussian $\mathcal{N}(x_t | m_t, P_t)$ with $m_t = x_{t-1} + c/(Q + R) [y_t - x_{t-1}]$, $P = Q - Q^2/(Q + R)$. If we replace the importance distribution with $\mathcal{N}(x_t | m_t, c P_t)$ where $c < 1$, then the weights become unbounded and thus the particle filter is not guaranteed to converge.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{Illustration of the effect of scaling the importance distributions with $c$ in linear Gaussian example. Left: Gaussian distribution. Right: Student’s $t$-distribution with $\nu = 3$ degrees of freedom. The scaling of variance significantly affects the performance of the particle filter with a Gaussian importance distribution whereas the effect to a Student’s $t$-distribution based particle filter is smaller.}
\end{figure}

\textbf{Example 4.2} (Non-linear state space model). A typically used example of a non-linear model is the following system (see, e.g., [21]):

$$
x_t = \frac{1}{2} x_{t-1} + 25 \frac{x_{t-1}}{1 + (x_{t-1})^2} + 8 \cos(1.2t) + q_{t-1}, \quad y_t = \frac{x_t^2}{20} + r_t,
$$

where $q_{t-1} \sim \mathcal{N}(0, 10)$ and $r_t \sim \mathcal{N}(0, 1)$. It is now easy to show that the ratio of the optimal importance distribution and any Gaussian distribution will be uniformly bounded. Thus using a non-linear Kalman filter based Gaussian importance distribution leads to a converging particle filter provided that we do not allow the Gaussian distribution to become singular.

\section{5. CONCLUSION}

In this paper, we extended the proof of Hu et al. (2008) [9] to the case of general importance distributions. Our proof shows the $L^4$ convergence of the particle filter estimates for a general class of unbounded functions provided that the importance weights are bounded. This also implies the probability-one convergence of the estimates [9]. We have analyzed the conditions set by the proof on importance distributions proposed in literature and tested them numerically.

\section{6. REFERENCES}


