ON THE DESIGN OF MATCHED ORTHONORMAL WAVELETS WITH COMPACT SUPPORT

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ABSTRACT

We develop new algorithms for designing matched wavelets and matched scaling functions using a new parametrization of compactly supported orthonormal wavelets that is developed in [4]. The parametrization specifies the wavelet filter coefficients as a continuous function of an arbitrary vector of half its length. Therefore, the wavelets and the scaling functions are also continuous functions of this vector through the use of the cascade algorithm. Hence, standard minimization procedures can be employed for wavelet design. In the case of matched scaling function we propose a new optimization algorithm that significantly reduces the problem complexity. We provide several design examples that illustrate the effectiveness of the proposed algorithm.

Index Terms—orthonormal wavelets, scaling function, optimization.

1. INTRODUCTION

Compactly supported orthonormal wavelets are an invaluable tool for signal modeling and analysis where the signal is decomposed using multiple scales of the mother wavelet. The classical design of compactly supported wavelets usually provides general-purpose wavelets that are useful for a wide-class of signals. For example, a wavelet with the maximum number of vanishing moments is useful in modeling signals that exhibit local polynomial-like behavior and it annihilates polynomials of order up to the number of vanishing moments. The construction of compactly supported orthonormal wavelets usually starts from the two-scale difference equation [1] or equivalently the multi-resolution analysis [2] to derive conditions on the wavelet filters coefficients to optimize the design objective. The resulting wavelets are in general independent of signal being analyzed.

In many applications, e.g., signal coding and compression, it is desirable to have maximum correlation between the underlying signal and the analysis wavelet or the corresponding scaling function. This would result in general to a sparse discrete wavelet transform and provides better understanding of the signal structure [3]. The signal-dependent wavelet design is referred to as matched wavelet. The wavelet decomposition with matched wavelet typically uses only one or two wavelet scales. Therefore, in some application scenarios it may be desirable to match the scaling function rather than the wavelet if the signal has more low frequency content.

In this work, we propose a new design procedure for matched wavelet and matched scaling function that is based on a new parametrization of compactly supported orthonormal wavelet [4]. The parametrization describes the wavelet filter coefficients as a continuous function of an arbitrary vector of half its length. The wavelet filter vector constitutes a basis for the null space of a special matrix that is parameterized by the decision vector. This parametrization along with the cascade algorithm [5] provide a continuous mapping between the decision variables and the wavelet or the scaling function. This allows for using gradient search techniques for minimizing the distance between the matched wavelet (or scaling function) and a template function. In the case of matched scaling function, we propose a new objective function that does not require the evaluation of the cascade algorithm and provides significant computational saving. The proposed algorithms are evaluated using several examples for the different optimization criteria.

The problem of designing matched wavelets has been studied earlier in the literature. In [6], an algorithm was developed to find the best approximation of a template signal up to a predefined number of scales by minimizing an upper bound of the error performance. In [3], a matched wavelet is designed to match a bandlimited signal in the frequency domain by matching both the spectrum magnitude and group delay. The resulting wavelet does not have compact support because of the bandlimited spectrum. In [7], several algorithms for designing biorthogonal and semi-orthogonal methods are proposed that are based on minimizing a cost function of the highpass filter to match the signal statistics as an estimator. There are many other approaches that are less relevant to our problem, e.g., using matching pursuit from a large dictionary [8], and designing biorthogonal basis using lifting [9]. Compared to these earlier approaches the proposed algorithm provides a new design procedure for wavelets that are both orthonormal and have compact support. The design is done in the time-domain and does not have assumptions on the template function. Further, the proposed framework encompasses the design of matched scaling function that is encountered in many practical applications.
2. BACKGROUND

Compactly supported orthogonal wavelets are usually computed using the two-scale difference equation of the scaling function $\phi(x)$ [1]

$$\phi(x) = \sqrt{2} \sum_{n=0}^{M-1} h_n \phi(2x - n) \quad (1)$$

and the scaling function is related to the mother wavelet $\psi(x)$ by

$$\psi(x) = \sqrt{2} \sum_{n=0}^{M-1} g_n \phi(2x - n) \quad (2)$$

The high-pass filter coefficients are usually chosen as [1]

$$g(n) = (-1)^{n+1} h(M - 1 - n) \quad (3)$$

Therefore, the wavelets is completely parameterized by the wavelet filter $\{h_n\}$. The computation of the scaling function and the wavelet function from the filter coefficients is usually done using the cascade algorithm [5] which iteratively computes the values of both function at a finer time grid. In general, $\phi(x)$ and $\psi(x)$ are continuous functions of $\{h_n\}$.

In [4], we showed that the vector, whose entries are the wavelet filter coefficients, constitute the null space of a special matrix $G$, of size $(M-1) \times M$ where $M$ is the length of the wavelet filter, that is parameterized by an arbitrary vector $(v, \sigma)$. If a column vector $p = [p_0, p_1, \ldots, p_M]'$, denote

$$\tilde{p} \triangleq [p_M \ - p_{M-1} \ p_{M-2} \ \cdots \ p_1 \ - p_0]' \quad (4)$$

Then the wavelet parameterization matrix could be written as [4]:

$$G(v, \sigma) = \left( \tilde{\gamma}_1 \ \gamma_1 \ \cdots \ \tilde{\gamma}_{M/2-1} \ \gamma_{M/2-1} \ \tilde{u} \right)’ \quad (5)$$

where

$$\tilde{u} \triangleq (1 \ - 1 \ \cdots \ 1 \ - 1)' \quad (6)$$

and $\{\gamma_i\}$ are vectors of length $M$ that are parameterized by the parameter vector $(v, \sigma) = (v_1 \ \cdots \ v_{M/2-1} \ \sigma)$. For $1 \leq i \leq M/2 - 1$ we have

$$\gamma_i \triangleq [0 \ v_{M/2-1} \ \cdots \ v_1 - \sigma v_1 \ \sigma v_2 \ \cdots \ - \sigma v_{2i-1} \ \sigma v_{2i}]’ \quad (7)$$

where it has $M - 2i$ zeros. Note that for $i \leq M/2$, $\gamma_i$ does not depend on $\sigma$. It was shown in [4], that if $G$ is full-rank then

$$Gh = 0 \quad (8)$$

where $h = [h(0) \ h(1) \ \ldots \ h(M-1)]'$. Note that, $h$ should be scaled to have unity norm. Further, $h$ can be computed as the projection of the vector $\tilde{u}$ of all ones onto on the null space of $G$, i.e.,

$$h = \frac{P_G^\perp u}{\|P_G^\perp u\|} \quad (9)$$

where $P_G^\perp$ is the orthogonal projection matrix onto the null space of $G$. A simple procedure for computing $P_G^\perp$ was described in [4]. From (9), the wavelet filter coefficients are continuous functions of the decision variables $(v, \sigma)$.

3. MATCHED WAVELETS

The purpose of designing a matched wavelet is to design a wavelet function $\psi(t)$ or a scaling function $\phi(t)$ so that it resembles a template function $f(t)$ under some distance measure. If the L2-norm is used then the objective is to minimize

$$J_1 = \|f(t) - \phi(t)\|^2 \quad \text{or} \quad \|f(t) - \psi(t)\|^2 \quad (10)$$

Note that, $\phi(t)$ and $\psi(t)$ are continuous functions of $\{h_n\}$ (from the two scale difference equation), and hence continuous in $(v, \sigma)$. It could be computed using the cascade algorithm [5]. Therefore, gradient search techniques could be used to solve for the optimal matched wavelet. However, the algorithm is rather complex because at each iteration, we need to invoke the iterative cascade algorithm for computing the scaling or the wavelet function.

If the objective is to match the scaling function, then another possible choice of the objective function that yields a suboptimal solution but significantly reduces the complexity is to minimize

$$J_2 = \|f(t) - \sqrt{2} \sum_n h_n f(2t - n)\|^2 \quad (11)$$

where $\{h_n\}$ are computed from (9). The rationale behind this objective function is that if $f(t)$ is a valid scaling function, then $J_2$ would be zero; otherwise, the scaling function that is generated using $\{h_n\}$ which minimizes $J_2$ would be closest to $f(t)$. The objective functions $J_1$ and $J_2$ are closely related. Consider the functional $\chi : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, where for $g(t) \in L^2(\mathbb{R})$

$$\chi(g) = g(t) - \sqrt{2} \sum_n h_n g(2t - n) \quad (12)$$

for a particular choice of the wavelet filter coefficients $\{h_n\}$. Clearly $\chi(\cdot)$ is linear, hence continuous [10]. Therefore, for each $\epsilon > 0$ there exist $\delta(\epsilon)$ such that if $\|g - f\| < \delta$, then

$$\|\chi(g) - \chi(f)\| < \epsilon.$$  

By choosing $g(t)$ as the scaling function $\phi(t)$, then $\chi(\phi) = 0$, and $\|\phi - f\|$ is equivalent to (10) while $\|\chi(\phi) - \chi(f)\|$ is equivalent to (11).

Further, $J_2$ in (11) can be bounded by scaling $J_1$ in (10). Note that, the scaling function $\phi(t)$ follows the two-scale difference equation (1), hence

$$f(t) - \phi(t) = \left( f(t) - \sqrt{2} \sum_n h_n f(2t - n) \right) + \sqrt{2} \sum_n h_n (f(2t - n) - \phi(2t - n)) \quad (13)$$
denote, $g(t) \triangleq f(t) - \varphi(t)$, then

$$g(t) - \sqrt{2} \sum_n h_n g(2t - n) = f(t) - \sqrt{2} \sum_n h_n f(2t - n)$$

(14)

By the triangle inequality [10], we have

$$\| f(t) - \sqrt{2} \sum_n h_n f(2t - n) \| \leq \| g(t) \| + \sqrt{2} \sum_n |h_n| \| g(2t - n) \|$$

(15)

Note that, $J_1 = \| g(t) \|^2 = 2 \| g(2t - n) \|^2$ and the left hand side is $J_2$. Hence, we have

$$J_2 \leq J_1 \left( 1 + \sum_n |h_n| \right)^2$$

(16)

In the following we show that, the objective function in (11) can be evaluated without invoking the iterative cascade algorithm. The objective function in $J_2$ could be written as

$$J_2 = \int \left( f(t) - \sqrt{2} \sum_n h_n f(2t - n) \right)^2$$

$$= \int |f(t)|^2 - 2 \sqrt{2} \sum_n h_n \int f(t) f(2t - n) +$$

$$2 \sum_n \sum_m h_n h_m \int f(2t - n) f(2t - m)$$

(17)

Then after removing the irrelevant terms, $J_2$ could be put in the equivalent matrix form,

$$J_2 = -b^t a + h^t B h$$

(18)

where

$$a_n \triangleq 2 \sqrt{2} \int f(t) f(2t - n)$$

(19)

$$b_{m,n} \triangleq 2 \int f(2t - n) f(2t - m)$$

(20)

Note that $B$ is symmetric. Then by incorporating (9) and discarding the irrelevant terms we get the equivalent form,

$$J_2 = - \frac{1}{\| P^G u \|} u^t P^G a + \frac{1}{\| P^G u \|^2} u^t P^G B P^G u$$

(21)

which can be easily solved using gradient search without a need to explicitly compute the scaling function at each iteration as in the first case.

Note that, because of the dyadic scaling in the orthogonal wavelet transform, we could preprocess the template function using up or down dyadic scaling. In the case of matched wavelets, this would result in matching with $\psi(2x)$ or $\psi(x/2)$ which are equally good as matching $\psi(x)$ for signal modeling using the wavelet transform.

4. DESIGN EXAMPLES

We show two examples for matching the scaling function and the mother wavelet. In both cases, we assume that the template function (to be matched) has a unity norm and has a vanishing moment at zero for matching wavelet, or unity dc component for matched scaling function.

First we evaluate the matching scaling function algorithm with the two objective functions (10) and (11). We use an exponentially damped cosine function as a template scaling function. The results are shown in Fig. 1. Note that, the two resulting solutions are similar especially in the passband.

The complexity of evaluating the objective function in (10) is several orders of magnitude more than the objective function in (11) because of the iterative cascade algorithm.

![Fig. 1](image-url) Example of matched scaling function with with filter order 14; opt1 refers to the solution of the objective function in (10), and opt2 refers to (11)
The following example illustrates a matched wavelet design by minimizing the objective function in (10) using gradient search. We use a template that is a windowed derivative of a sinc function. The resulting matched wavelet is illustrated in Fig. 2.

![Wavelet function and frequency magnitude](image)

**Fig. 2.** Example of matched wavelet function with a template of a sinc derivative and filter order 18

5. CONCLUSION

We developed a framework for designing compactly supported matched orthonormal wavelet and scaling function. The framework is based on the matrix parametrization of the wavelet filter coefficients where filter coefficients are expressed as a continuous function of the decision variables. This enables the use of standard gradient algorithms for optimization. The number of decision variables is half the length of the wavelet filter. In case of matched scaling function, we proposed a suboptimal objective function that significantly reduces the design complexity with bounded degradation in the performance. The effectiveness of the proposed algorithms was established using several design examples. The proposed algorithm has an advantage over earlier approaches, that it always yields compactly supported orthonormal wavelets or scaling functions with no assumptions on the template function. A similar procedure is used in [4] to derive a framework for general wavelet design where classical wavelets are explicitly computed.

6. REFERENCES