ADJUSTABLE BANDWIDTH FILTER DESIGN WITH GENERALIZED FARROW STRUCTURE

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ABSTRACT

Digital filters with adjustable bandwidth(s) are generally desirable in many applications like audio processing and telecommunication. This paper proposes a generalized Farrow structure for adjustable bandwidth linear-phase FIR filters designed under a minimax design criterion. The bandwidth of the proposed filter structure can be continuously adjusted with an updating routine that only involves a few multiplications and additions. Moreover, the generalized structure can be designed to effectively reduce the dynamic range of basis filter coefficients, which is desirable when making a fixed-point implementation on FPGAs.

Index Terms— adjustable bandwidth filter, Farrow structure, minimax

1. INTRODUCTION

Many applications such as digital audio, telecommunication and instrumentation usually require digital filters with adjustable bandwidth(s) while the phase response is kept linear. A transformation-based network technique was proposed in [1] for implementing variable-cutoff linear-phase FIR filters. A least-squares design technique [2] has also been considered. In this paper, a special filter structure, called the Farrow structure[3] is exploited and generalized in the design and implementation of adjustable bandwidth filters.

Conventionally, to realize adjustable bandwidth filter, the bandwidth range of interest is first discretized into \( N \) discrete bandwidths. A specific filter design routine, e.g., windowing or Parks-McClellan[4], is employed to design a family of filters—one for every individual bandwidth. All these \( N \) sets of filter coefficients are then stored in memory and invoked on demand. A significant drawback of this scheme is that the memory occupation is dependent on the discretization granularity of the bandwidth range. The bandwidth is also restricted to be adjusted in a discrete manner. For comparison, the Farrow structure is composed of a bank of fixed linear-phase FIR basis filters, which are designed off-line and optimized for a given range of bandwidths. The basis filters are then combined online in such a way[5] that the bandwidth(s) of the overall filter can be continuously adjusted by one or two scalar bandwidth parameter(s). The Farrow structure was originally invented for the design of fractional delay FIR filters[3]. A modified version of it has been proposed in [6] which reduced the overall arithmetic complexity. The application of Farrow structure has also been extended to adjustable multirate filter design[7].

A key problem with the original Farrow structure is that the basis filter coefficient magnitudes increases exponentially with the order of the basis filters, which is difficult to handle when a fixed-point representation is needed because the filter coefficients have a large dynamic range. To overcome this problem, a generalized Farrow structure and accompanying design algorithm are proposed to reduce the coefficients dynamic range while maintaining the same performance. Although this paper concentrates on the implementation of adjustable bandwidth filter with the generalized Farrow structure, the generalized structure can also be applied to adjustable multirate or fraction delay filter design problems.

Section 2 formulates the linear-phase FIR filter design under the minimax criterion into a standard linear programming(LP) problem. Section 3 generalizes the original Farrow structure and recasts the basis filter design algorithm as a larger scale linear program based on the formulation in Section 2. Numerical examples are then shown to justify the feasibility and superiority of the proposed filter structure.

2. MINIMAX FIR FILTER DESIGN BY LINEAR PROGRAMMING

Take low pass type I FIR filter design as an example. As shown in Fig. 1, the filter specifications are divided into 3 bands: pass band, transition band and stop band. The designed frequency responses under pass band and stop band are bounded by a maximum ripple magnitude. The minimax criterion aims to minimize the maximum ripple magnitude in both bands. Denote the filter length to be \( N(\text{odd}) \) and set \( M = (N + 1)/2 \). Positive symmetry indicates that the filter impulse response satisfies

\[
h(m) = h(N - 1 - m), \quad m = 0 \cdots M - 2 \quad (1)
\]
According to [4], the frequency response of $h$ can be expressed as

$$H(e^{j\omega}) = \sum_{k=0}^{N-1} h(k)e^{-jkw} = e^{-j\omega(N-1)/2}H_R(\omega)$$  \hspace{1cm} (2)$$

Where

$$H_R(\omega) = \sum_{m=0}^{M-1} a(m) \cos(m\omega)$$  \hspace{1cm} (3)$$

And

$$a(0) = h(M-1)$$
$$a(m) = 2h(M-1-m) \quad m = 1 \cdots M-1$$  \hspace{1cm} (4)$$

Denote

$$\mathbf{a} = [a(0) \ a(1) \ \cdots \ a(M-1)]^T$$  \hspace{1cm} (5)$$

For low pass filters, denote the pass band edge as $\omega_p$ and stop band edge as $\omega_s$, where $\omega_p < \omega_s$. The frequency band of interest $[0, \omega_p] \cup [\omega_s, \pi]$ can be discretized into $N_p$ and $N_s$ grids respectively. Denote the discretized frequency vector as

$$\omega = [0 \ \cdots \ \omega_p \ \omega_s \ \cdots \ \pi]^T \in R^{N_p+N_s}$$

The desired response under frequency grid $\omega$ to be

$$\mathbf{D} = [1 \ \cdots \ 1 | 0 \ \cdots \ 0]^T \in R^{N_s}$$  \hspace{1cm} (7)$$

The designed frequency responses are defined as $\mathbf{H} = [H_R(e^{j\omega_1}) \ \cdots \ H_R(e^{j\omega_{N_p}})]$. Denote the maximum approximation ripple under $\omega$ to be $\delta = [\delta_p \ \cdots \ \delta_p | \delta_s \ \cdots \ \delta_s]$, where $\delta_p$ and $\delta_s$ stands for the pass band and stop band ripple.

Fix the ratio between $\delta_p$ and $\delta_s$ to be $\frac{\delta_p}{\delta_s} = r$, $\delta$ and $r$ can be expressed as

$$\delta = \delta_s \left[ r \cdots r | 1 \cdots 1 \right]^T$$  \hspace{1cm} (8)$$

(3) can be rewritten as

$$\mathbf{H} = \mathbf{Ca}$$  \hspace{1cm} (9)$$

The ripple constraints become

$$|\mathbf{H} - \mathbf{D}| \leq \delta$$  \hspace{1cm} (10)$$

Combing with (9)(8), we have

$$\left[ \begin{array}{cc} \mathbf{C} & \mathbf{r} \\ -\mathbf{C} & \mathbf{r} \end{array} \right] \mathbf{a} - \left[ \begin{array}{c} \delta \\ \delta \end{array} \right] \leq \left[ \begin{array}{c} \mathbf{D} \\ -\mathbf{D} \end{array} \right]$$  \hspace{1cm} (11)$$

Define $\mathbf{A}$, $\mathbf{x}$, $\mathbf{d}$ as follows,

$$\left[ \begin{array}{ccc} \mathbf{C} & \mathbf{r} & \mathbf{a} - \delta_s \\ -\mathbf{C} & \mathbf{r} & \mathbf{d} \end{array} \right] \leq \left[ \begin{array}{ccc} \mathbf{D} & -\mathbf{D} \end{array} \right]$$  \hspace{1cm} (12)$$

Here, $\mathbf{x} \in R^{M+1}$ is the design vector. Define the cost vector to be $\mathbf{c} = [0 \ \cdots \ 0 \ -1] \in R^{M+1}$ so that $\mathbf{c}^T \mathbf{x} = \delta_s$. Therefore, the filter design problem that aims to minimize the maximum approximation ripple can be recast as the following linear programming (LP) problem:

$$\min \ \mathbf{c}^T \mathbf{x} \quad s.t. \quad \mathbf{Ax} \leq \mathbf{d}$$  \hspace{1cm} (13)$$

The LP has $M + 1$ variables and $2N_g$ constraints, which can be solved in polynomial time using an interior point method.

3. GENERALIZED FARROW STRUCTURE

![A generalized Farrow structured adjustable bandwidth filter with bank order $L$](image)

The original Farrow structure is composed of several parallel linear-phase FIR basis filters. The output of the $i$-th basis filter $H_i(z)$ is scaled by a monomial $(x^i)$ of an adjustable parameter. For the adjustable bandwidth filter structures, the parameter is the bandwidth $b$ offset by a constant $b_0$[5]. In case of adjustable fractional delay filter structures, the parameter is simply the fraction factor[6]. The output of each channel is finally summed up to yield the output. This paper generalizes the basis filter scaler to any orthogonal polynomial set.
\{p_i(x)\}, and then shows that the dynamic range of the designed basis filter coefficients can be reduced significantly by choosing a proper polynomial set.

Taking an adjustable low pass filter design problem as an example, the basis filters \(H_0(z)\) through \(H_L(z)\) are all assumed to be type I FIR filters with filter design parameter vectors \(a_0\) to \(a_L\). Suppose the bandwidth parameter \(b\) is specified to change continuously in the range \([b_l, b_u]\) with a transition band \(\Delta b\). Then \(b_0\) is set as \((b_l + b_u)/2\) such that \(b - b_0 \in [-((b_u - b_l)/2), (b_u - b_l)/2]\). In the design process, \(b\) is first discretized into \(N_d\) values in \([b_l, b_u]\). For a specific discretized \(b_i\), the pass band and stop band edge is set as \(\omega_p = b\) and \(\omega_s = b + \Delta b\). We can define \(A_i\) and \(d_i\) in the same manner as (12) for each \(b_i\). The equivalent overall filter design vector \(\bar{\alpha}\) for each \(b_i\) can be expressed as a linear combination of \(a_0\) through \(a_L\).

Define
\[
P_{ik} = \text{diag}(\{p_k(b_i - b_0) \cdots p_k(b_i - b_0)\});
\] (14)
The equivalent overall filter design vector \(\bar{\alpha}_i\) is expressed as
\[
\bar{\alpha}_i = [P_{i0} \cdots P_{iL}] \begin{bmatrix} a_0 \\ \vdots \\ a_L \end{bmatrix}
\] (15)
The ripple constraints for \(b_i\) become
\[
A_i \begin{bmatrix} \bar{\alpha}_i \\ -\delta_s \end{bmatrix} \leq d_i
\] (16)
Define
\[
P_i = \begin{bmatrix} P_{i0} & \cdots & P_{iL} & 0 \\ 0^T & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}
\] (17)
The overall design vector is defined as
\[
\bar{x} = [a_0^T \cdots a_L^T - \delta_s]^T;
\] (18)
The ripple constraints finally become
\[
A_i P_i \bar{x} \leq d_i
\] (19)
Stacking all \(N_d\) discretized bandwidth \(b_i\) inequalities together, we have
\[
\begin{bmatrix} A_1 P_1 \\ \vdots \\ A_{N_d} P_{N_d} \end{bmatrix} \bar{x} \leq \begin{bmatrix} d_1 \\ \vdots \\ d_{N_d} \end{bmatrix}
\] (20)
Here \(\bar{x} \in R^{(L+1)M+1}, \bar{\alpha} \in R^{2N_d \times (L+1)M+1}, \bar{d} \in R^{2N_d \times N_d}\). Define the overall cost vector as
\[
\bar{c} = [0 \cdots 0 - 1] \in R^{(L+1)M+1}. \quad \text{The basis filters can be designed by running an LP taking the same form as shown in (13).}
\]
It’s worthwhile pointing out that as long as the discretization number \(N_d\) is large enough, \(b\) can be adjusted continuously rather than in discrete steps following \(b_i\) when synthesizing the basis filters as shown in Fig. 2.

Figure 3 shows an example of the frequency response of an adjustable low pass filter under Chebyshev polynomial scaling. Although \(b\) is only discretized into \(N_d = 5\) steps, the frequency response is well behaved using bandwidths other than the five optimized ones. It has been observed that

![Figure 3](image-url)

**Fig. 3.** Frequency response of a variable low pass filter designed by Chebyshev polynomial scaling. Design parameters are \(N = 25, L = 4, N_d = 5, N_d = 433, b_l = 0.05\pi, b_u = 0.25\pi, \Delta b = 0.2\pi, r = 1\), the resulting \(\delta_s = 5.53 \times 10^{-3}\).

![Figure 4](image-url)

**Fig. 4.** Basis filter coefficients histogram comparison between monomial and Chebyshev type I polynomial scaling

the choice of different family of polynomials for scaling has little influence on the equivalent overall filter performance. However, the magnitude of the basis filter coefficients can vary significantly under different polynomial scalers. Fig. 4 shows the histogram of all the coefficients of the basis filters designed by monomial scaling and Chebyshev polynomial scaling. The dynamic range of the coefficients in monomial scaling basis filters is twice as wide as that in Chebyshev polynomial scaling case, suggesting that 1-bit word length can be saved in the latter case to effectively represent all the coefficients.

Figure 5 shows how the maximum ripple magnitude of
the frequency response varies with respect to coefficient word length for the quantized basis filters synthesized by different scaling polynomials. The maximum ripple of monomial scaling filter structure and Chebyshev polynomial scaling structure is the same without quantizing the basis filter coefficients (shown as the floating point line). However, when quantized to the same word length, the ripple magnitude of Chebyshev polynomial scaling filter structure is smaller than that of monomial scaling. The maximum difference could be as much as 10 dB for 21-bit word length in Fig. 5. As more bits are used, the difference between two designs becomes smaller and converges to the floating-point case.

![Figure 5](image-url)

**Fig. 5.** Maximum ripple magnitude comparison between quantized filters designed by monomial and Chebyshev polynomial scaling


Figure 6(a) shows that for a fixed value of polynomial magnitude decreases exponentially as polynomial monomial scaler is given by

\[ T_i(x) = \cos(i \arccos(x)) \]  

(22)

In Fig. 6(b), for a fixed value of \( b \) other than \( b_0 \), the polynomial magnitude is not proportional monotonic with respect to the index \( i \). As a result, the designed basis filter coefficient magnitudes are more balanced and evenly distributed over different indices. Other orthogonal polynomials like Legendre and Hermitian polynomials also show similar characteristics as Chebyshev polynomial.

4. CONCLUSION

This paper proposed a generalized Farrow structure for the design of adjustable bandwidth filters. The filter design process is formulated into a standard linear programming problem that can be solved in polynomial time. When fixed-point arithmetic is considered, the proposed structure is able to perform better than the original Farrow structure by selecting an appropriate polynomial scaling function. Although this paper concentrates on the case of adjustable low pass filter design, the application of the generalized Farrow structure can be easily extended to adjustable high pass, band pass and band reject filter design problems, as well as adjustable fractional delay and multirate filters.

5. REFERENCES


