ABSTRACT
In this work, the tracking analysis of the Normalized Least Mean Fourth (NLMF) algorithm is investigated for a random walk channel under very weak assumptions. The novelty of this work resides in the fact that no restrictions are made on the dependence between the input successive regressors, the dependence among input regressor elements, the length of the adaptive filter, the distribution of noise and filter’s input. Moreover, in our approach, there is no restriction made on the step size value and therefore the analysis holds for all the values of the step size in the range of stable NLMF algorithm. The analysis is based on a recently proposed performance measure called effective weight deviation vector which is the component of weight deviation vector in the direction of input regressor. In this paper, asymptotic time-averaged convergence for the mean square effective weight deviation, mean absolute excess estimation error, and mean square excess estimation error for the NLMF algorithm is established. Finally, a number of simulation results are carried out to corroborate the theoretical findings.

Index Terms — Adaptive filters, NLMF algorithm, Convergence Analysis.

1. INTRODUCTION
The Normalized Least Mean Fourth (NLMF) algorithm [1] is the normalized version of the Least Mean Fourth (LMF) algorithm [2]. The analysis of the NLMF becomes difficult because of the normalization term. Therefore, until now, the analysis of the NLMF algorithm is carried out using some strong assumptions [3, 4], for example, using the independence assumption [5], long filter assumption [6]. Recently a new performance measure, the effective weight deviation vector, is introduced for the convergence analysis of the NLMS algorithm [7]. This vector is the component of weight deviation vector in the direction of input regressor vector. It is shown that the effective weight deviation is the only component that contributes to the excess estimation error [7]. Therefore, the analysis based on the study of this component can give more insight on the performance of the adaptive algorithm. In this paper, we have used the framework of [7] for the tracking analysis of the NLMF algorithm.

The main contribution of this paper is a rigorous tracking analysis of the NLMF algorithm that has the following advantages: (1) It holds for arbitrary dependence among successive regressor vectors, (2) It holds for arbitrary dependence among the elements of regressor vector, (3) This analysis is not restricted to the class of long filters, (4) It holds for arbitrary distributions of the filter input and the noise, and (5) It holds for all the values of the step size in the range that ensures the stability of the NLMF algorithm.

The paper is organized as follows. In Section 2, the framework of our analysis along with a brief overview of the newly introduced performance measure is presented. In Section 3, asymptotic time-averaged tracking analysis for the mean square effective weight deviation, mean absolute excess estimation error, and the mean square excess estimation error of the NLMF algorithm is carried out. Simulation results are presented to validate the theoretical findings in Section 4 and paper is ended with concluding remarks in Section 5.

2. OUR APPROACH
Our approach to analyze the NLMF algorithm is based on a recently proposed performance measure called effective weight deviation vector which was originally introduced for the convergence analysis of the NLMS algorithm [7]. Before presenting the brief overview of the effective weight deviation vector, system model used for the analysis is described in the ensuing section.

2.1. System Model
Consider the scenario of tracking a random walk channel. The desired response \(d_k\) for the adaptive filter is obtained from the output of the plant, i.e.,
\[
d_k = c_k^T x_k + \eta_k, \tag{1}
\]
where
\[
c_k = [c_{1,k}, c_{2,k}, \ldots, c_{N,k}]^T \tag{2}
\]
is a time-varying plant vector according to random walk model
\[
c_{k+1} = c_k + q_k, \tag{3}
\]
with \(q_k\) is a random vector of same dimension as \(c_k\) and
\[
x_k = [x_{1,k}, x_{2,k}, \ldots, x_{N,k}]^T \tag{4}
\]
is the input data vector at time \(k\). \(\eta_k\) is the plant noise, \(N\) is the number of plant parameters, and \([\cdot]^T\) is the transpose operation. The tracking error \(e_k\) is defined as
\[
e_k = d_k - w_k^T x_k, \tag{5}
\]
The weight update recursion of the NLMF algorithm [1] is described by
\[
w_{k+1} = w_k + \mu c_k x_k / ||x_k||^2, \tag{6}
\]
where \(\mu > 0\) is the algorithm step size and the norm of a vector \(x\) is defined as \(||x|| \equiv \sqrt{x^T x}\). The error \(e_k\) can be decomposed...
to two terms: the plant noise $\eta_k$ and the excess estimation error $\varepsilon_k$ defined by

$$\varepsilon_k = e_k - \eta_k. \quad (7)$$

$\varepsilon_k$ is also termed as adaptation noise [8] since it represents the noise that appears at the filter output due to adaptation. The signal behavior of adaptive filter is described by the evolution of the moment of $\varepsilon_k$ with the time. The weight deviation vector is defined by

$$v_k = w_k - c_k. \quad (8)$$

due to (1), (5), (7), and (8), it can be shown that

$$\varepsilon_k = -v_k^T x_k. \quad (9)$$

2.2. Review of Effective Weight Deviation: A New Performance Measure

In this section, a brief overview of effective weight deviation vector [7] is presented. Let $u_k$ denote a unit vector along the direction of the vector $x_k$, i.e.,

$$u_k = \begin{cases} x_k/|x_k| & \text{if } x_k \neq 0 \\ \text{an arbitrary unit vector} & \text{if } x_k = 0 \end{cases}$$

Consequently, the weight deviation vector $v_k$ can be decomposed into two orthogonal components; the first component $\tilde{v}_k$ is the projection of $v_k$ along the direction of vector $x_k$, while the second component $\hat{v}_k$ is orthogonal to $x_k$. The vectors $\tilde{v}_k$ and $\hat{v}_k$ are given, respectively, by

$$\tilde{v}_k = (u_k^Tv_k)u_k, \quad (10)$$

$$\hat{v}_k = v_k - \tilde{v}_k. \quad (11)$$

Due to unit vector $u_k$ and (10), the vector $\tilde{v}_k$ satisfies

$$\tilde{v}_k^T x_k = v_k^T x_k = -\varepsilon_k, \quad (12)$$

Equations (11), (12), and (9) imply that

$$\tilde{v}_k^T x_k = v_k^T x_k = -\varepsilon_k \quad (13)$$

Ultimately, it can be shown that

$$\tilde{v}_k^T x_k = 0. \quad (14)$$

Thus, only the component $\tilde{v}_k$ contributes to the excess estimation error. The reminder, $\hat{v}_k$, of the weight deviation vector $v_k$ does not contribute to the excess estimation error. For this reason, $\tilde{v}_k$ is called as “the effective weight deviation vector” [7]. From (10) and (13), it can be shown that

$$|\varepsilon_k| = ||\tilde{v}_k|| ||x_k||. \quad (15)$$

Equation (15) shows that what matters in determining the magnitude of excess estimation error is the length of vector $\tilde{v}_k$ rather than the length of $v_k$. Thus, studying the behavior of $||\tilde{v}_k||$ gives a generally brighter insight on the perform of the algorithm than studying the behavior of $||v_k||$. The theoretical advantage of $\tilde{v}_k$ in the context of the NLMF algorithm is that it can be analyzed without the need to calculate mathematical expectations of quantities normalized by $||x_k||^2$. This is due to the fact that the normalization by $||x_k||^2$ already included in the definition of $\tilde{v}_k$, as seen by (12). Therefore, $\tilde{v}_k$ enables a rigorous analysis of the NLMF algorithm under weak assumptions. In this work, we derived an upper bound on the long term average of mean square effective weight deviation ($E[|v_k|^2]$), mean square excess estimation error ($E[\varepsilon_k^2]$), and mean absolute excess estimation error ($E[|\varepsilon_k|]$). These long term averages are defined as follows:

$$L_1 \triangleq \operatorname{Limsup}_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} E[||\tilde{v}_j||^2], \quad (16)$$

$$L_2 \triangleq \operatorname{Limsup}_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} E[\varepsilon_j^2], \quad (17)$$

$$L_3 \triangleq \operatorname{Limsup}_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} E[|\varepsilon_j|], \quad (18)$$

where the notation “Limsup” is defined by

$$\operatorname{Limsup}_{k \to \infty} s_k \equiv \operatorname{Lim}_{k \to \infty} \left( \sup_{i \geq k} s_i \right), \quad (19)$$

where “sup” means supremum which refers to least upper bound. The smaller is the value of long term averages (16), (17), and (18), the finer is the steady state performance of algorithm and vice versa. The upper bound of the long term average (16) is used along with (15) to derive bounds for mean square excess estimation error and mean absolute excess estimation error.

3. TRACKING ANALYSIS OF THE NLMF ALGORITHM

In this section, the tracking analysis of the NLMF algorithm is carried out with stationary input signal and plant for a random walk channel. Following are the assumptions employed in the analysis:

A1 : Sequences $\{x_k\}$, $\{q_k\}$, and $\{\eta_k\}$ are mutually independent.

A2 : Sequence $\{x_k\}$ is stationary with finite $E[|x_k|^2]$.

A3 : Sequence $\{\eta_k\}$ is a stationary sequence of independent zero mean random variables with finite even moments (i.e., with finite $E[\eta_k^2] = \sigma_\eta^2$, $E[\eta_k^4] = \phi_4$, $E[\eta_k^6] = \phi_6$, etc.).

A4 : The sequence $\{q_k\}$ is a stationary sequence of independent zero mean random vectors with finite second moments.

Assumption A2 can be well justified as in the case of NLMS algorithm [7], assumption A3 is very common and true in many practical cases while assumptions A1 and A4 are commonly used in analyzing the tracking of a random walk channel [5].

3.1. Analysis of Effective Weight Deviation Vector

The update recursion for the weight deviation vector $(\tilde{v}_k)$ is obtained using (1), (3), (5), and (6) and found to be:

$$v_{k+1} = v_k + \mu (\eta_k - v_k^T x_k)^3 \frac{x_k}{||x_k||^2} - q_k. \quad (20)$$

As we are going to derive the upper bound for steady-state scenario, the higher order terms of $v_k^T x_k$ can be ignored (since excess estimation error is very small at steady-state) and we can use the following approximation:

$$v_{k+1} \approx v_k + \mu (\eta_k^3 - 3\eta_k^2 v_k^T x_k) \frac{x_k}{||x_k||^2} - q_k. \quad (21)$$

Upon squaring both sides of the above equation and using assumptions A1-A4 and (8), it is found that:

$$E[|v_{k+1}|^2] \approx E[|v_k|^2] - (6\mu \sigma_\eta^2 + 9\mu^2 \phi_4) E[||\tilde{v}_k||^2] + \mu^2 \phi_6 E\left[\frac{1}{||x_k||^2}\right] + E[||q_k||^2]. \quad (22)$$
Since $E[|v_{k+1}|^2]$ is a positive quantity, therefore iterating the above equation backward $(k-1)$ iterations and then dividing the recursion by $k$, one obtains

$$0 \leq \frac{1}{k} E[|v_1|^2] - (6\mu^2 - 9\mu^2\phi_0^2)E \left[ \sum_{j=1}^{k} E \left[ ||v_j||^2 \right] \right] + \mu^2\phi_0^2 E \left[ \frac{1}{|x_1|^2} \right] + E \left[ ||q_1||^2 \right].$$  

Finally, by taking the limit as $k \to \infty$ and using the definition (16), the following upper bound can be provided:

$$L_1 \leq \frac{1}{(6\sigma^2_q - 9\mu\phi_0^2)} \left[ \mu\phi_0^2 E \left[ \frac{1}{|x_1|^2} \right] + \mu^{-1} E \left[ ||q_1||^2 \right] \right].$$  

(24)

It is worthwhile to note that we are not restricting our tracking analysis for slow plant variation which is commonly employed in the analysis of adaptive filtering. Moreover, the bound given by (24) has all the points of strength which are mentioned in Section 1. Furthermore, unlike the constant plant case, the bound for tracking random walk plant (24) is not a monotonically increasing function of step-size. In contrast, the first term in the bracket (i.e., $\mu\phi_0^2 E \left[ \frac{1}{|x_1|^2} \right]$) is increasing with step-size while the second term (i.e., $\mu^{-1} E \left[ ||q_1||^2 \right]$) is decreasing with step-size.

3.2. Analysis of Excess Estimation Error

The cases of bounded and unbounded plant input are analyzed separately.

Case 1: Bounded Plant Input

To proceed the analysis with bounded plant input, we use following assumption:

A6: There exist a positive number $B$ such that $|x_k| < B$ for all $k$.

This assumption is valid in many practical cases as naturally input data is bounded. Consequently, using A1 - A5, it can be shown that bound on the long term average of mean-square excess estimation error is

$$L_2 \leq \frac{B^2}{(6\sigma^2_q - 9\mu\phi_0^2)} \left[ \mu\phi_0^2 E \left[ \frac{1}{|x_1|^2} \right] + \mu^{-1} E \left[ ||q_1||^2 \right] \right].$$  

(25)

Case 2: Unbounded Plant Input

In this case too, we need the following assumption to simplify the analysis:

A6: The sequence $\{x_k\}$ is stationary with finite $E[|x_k|^2]$.

This is a weak assumption as the second order moment of input regressor generally exist. As a result, by employing A1 - A4 and A6, bound on the long term average of absolute excess estimation error can be shown to be:

$$L_3 \leq \frac{E \left[ ||x_1||^2 \right]}{(6\sigma^2_q - 9\mu\phi_0^2)} \left[ \mu\phi_0^2 E \left[ \frac{1}{|x_1|^2} \right] + \mu^{-1} E \left[ ||q_1||^2 \right] \right].$$  

(26)

The analytical results of the tracking analysis of the NLMF algorithm are summarized in the following table which demonstrate bounds on $L_1$, $L_2$, and $L_3$.

4. SIMULATION RESULTS

In this section, the performance analysis of the NLMF algorithm is investigated in an unknown system identification problem with

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Upper Bound</th>
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<tbody>
<tr>
<td>$L_1$</td>
<td>$\frac{1}{(6\sigma^2_q - 9\mu\phi_0^2)} \mu\phi_0^2 E \left[ \frac{1}{</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$\frac{B^2}{(6\sigma^2_q - 9\mu\phi_0^2)} \mu\phi_0^2 E \left[ \frac{1}{</td>
</tr>
<tr>
<td>$L_3$</td>
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(c = [1, 1, … , 1]T). System noise $\eta_k$ is a zero mean i.i.d. sequence with variance 0.01. The plant input regressor vector $x_k = [x_k, x_{k-1}, … , x_{k-N+1}]^T$ with $x_k$ being stationary zero mean unity variance. The objective of our simulations is to validate the derived analytical results without restrictions on

1. The dependence between successive regressors,
2. The dependence between the components of regressor,
3. The value of step-size in the range $0 < \mu < \frac{1}{1 + 3\sigma_q^2}$ [3],
4. The length of adaptive filter,
5. The distribution of the filter input and the noise.

To meet the above mentioned objectives, simulation experiments are carried out for the tracking a random walk channel. The first two objectives are achieved via generating a highly correlated input sequence as follows:

$$x_k = \beta x_{k-1} + \sqrt{1 - \beta^2} w_k$$  

(27)

where $\beta$ is a correlation factor and $w_k$ is a zero mean unity variance i.i.d. sequence. In our simulations, we have used $\beta = 0.95$ showing a highly correlated input sequence. In order to show that our analytical results holds for all the values of the step size in the range of stable NLMF algorithm, all simulation experiments are
carried out for a wide range \([0.01, 1]\) of step-size. A Random walk channel is generated using model (3).

Figure 1 depicts the comparison between both the theoretical and simulation results of long term average of mean-squared effective weight deviation for two different values of \(\sigma_q^2\), i.e., \(10^{-6}\) and \(10^{-7}\). The filter length used is 4 while both input and noise are uniformly distributed. As can be seen in Fig. 1, close agreement between theory and simulation is obtained. It is observed from this figure that degradation in performance is obtained by increasing the value of \(\sigma_q^2\). Also, unlike in the constant channel case, the long term average of mean-squared effective weight deviation is a not a monotonically decreasing function of the step size. In Fig. 2, long term average of mean-squared effective weight deviation is investigated with a filter length equal to 20 (which corresponds to a longer adaptive filter) while both input and noise are Gaussian distributed and \(\sigma_q^2 = 10^{-7}\). Here too, similar behavior is observed showing that our analytical results are not dependent on the adaptive filter’s length and on the distribution of the filter input and the noise.

Finally, the long term average of mean-squared excess estimation error for tracking random walk channel with stationary uniform input and \(\sigma_q^2 = 10^{-7}\) is investigated in Fig. 3 with filter length equal to 4. Again, consistency in the results is obtained which further validates our analytical findings.

5. CONCLUSION

In this work, a rigorous tracking analysis of the NLMF algorithm is carried out for a random walk channel using a newly proposed performance measure called effective weight deviation vector. The analysis is rigorous in a sense that it does not restrict to any limitations on the dependence between input successive regressors, the dependence among input regressor elements, the length of the adaptive filter, the distribution of noise and filter’s input, the value of step-size. Asymptotic time-averaged convergence for the mean square effective weight deviation, mean absolute excess estimation error, and the mean square excess estimation error for the NLMF algorithm is performed and consequently new explicit upper bounds for the long term average of mean-squared effective weight deviation, mean-squared excess estimation error, and mean absolute excess estimation error. Simulation results verified our theoretical findings.

Acknowledgments: This research work is funded by King Fahd University of Petroleum & Minerals (KFUPM) under Research Grant (FT080005).

6. REFERENCES