AMBIGUITY FUNCTIONS OF COMPRESSIVELY SENSED AND PROCESSED RADAR WAVEFORMS

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ABSTRACT

Compressive sensing and processing delivers high resolution data using reduced sampling rates and computational effort compared to Nyquist sensing and processing. Compressive processing, however, results to an increase in estimation error, which is particularly high when using a non-adaptive compressive sensing scheme. In this work, an adaptive compressive acquisition method is proposed for target tracking that utilizes information on target state available from the sequential tracking process. The proposed method reduces the peak ambiguity function sidelobe level compared to a non-adaptive method, thus improving tracking performance while adding little complexity to the compressive receiver.

Index Terms— Ambiguity function, compressive sensing and processing

1. INTRODUCTION

A highly concentrated ambiguity function (AF) [1] that also lies on a fine delay-Doppler grid reduces uncertainty in target state. In conventional acquisition and processing, measurements are sampled at the Nyquist rate followed by matched filtering (MF). AF concentration then depends on the type of the transmitted waveform while resolution in delay-Doppler depends on the length of the transmitted sequence and its bandwidth [2]. Nyquist sensing and processing (NSP) of high resolution measurements, however, requires a high analog to digital conversion rate and intensive processing of high dimensional sequences.

Compressive sensing and processing (CSP) [2, 3] can be used instead of NSP with reduced sampling rates and processing loads. Compressive sensing samples signals, that are sparse in some basis or dictionary, well below the Nyquist rate with no loss of information. The process is equivalent to projecting a high-dimensional waveform onto an acquisition matrix, that is incoherent with the sparsifying basis or dictionary, producing a low-dimensional sequence. A non-adaptive random acquisition matrix simplifies the receiver and captures information in any type of waveform [4–6]. Compressive processing [2, 3] operates on the low dimensional compressively sensed sequence with reduced computational effort. The drawback of CSP is that it introduces an additional estimation error [3]. In radar, this error is due to an increase in the compressed AF sidelobes and a reduction in the SNR at the output of the compressed MF [2] compared to when using NSP. The increase in error is particularly high when using a non-adaptive random compressive acquisition matrix.

In this work, a novel acquisition matrix is introduced for compressive sensing that incorporates sequentially arriving information on the target state from a particle filter based tracker [2, 7]. The problem setup and proposed method differ from existing work on adaptive compressive sensing methods, such as [8], which adapt the compressive sampling process according to previous sampling steps. Instead, the sensing mechanism is adapted prior to a single step compressive sampling process. The proposed method adds little complexity to the compressive sensing receiver by using information readily available from the tracking algorithm and operating at low data acquisition and processing rates. Theoretical results and numerical evidence are provided demonstrating a reduction in the peak sidelobe level of the compressed AF when using the adaptive matrix versus when using a random matrix. Moreover, in a companion paper [9] an improvement in the SNR at the output of the compressed MF and in tracking performance is reported when using the adaptive versus the random matrix.

2. NYQUIST AMBIGUITY FUNCTIONS

The two radar waveforms used in this work are a Björck constant amplitude autocorrelation (CAZAC) sequence [7, 10] and a multicarrier phase-coded (MCPC) [1] sequence. Björck CAZAC sequences are associated with AFs that are highly localized, with very low sidelobes. MCPC waveforms [1] are constructed by adding multiple waveforms that are modulated by orthogonal carriers separated in frequency using orthogonal frequency division multiplexing as:

\[ s(m) = \sum_{n=0}^{\beta-1} g_n(\left\lfloor \frac{m}{\beta} \right\rfloor) e^{-j2\pi \frac{nm}{\beta}} \] (1)

where \( m = 0, \ldots, \alpha\beta - 1 \) and \( \lfloor . \rfloor \) denotes rounding down.
to the nearest integer. Moreover, \( g_{n}(m), m = 0, \ldots, \alpha - 1, \ n = 0, \ldots, \beta - 1 \) represent the \( \beta \) sequences in \( s(m) \) of length \( \alpha \), that can be of any type. Therefore, the MCPC sequence \( s(m) \) has length \( M = \alpha \beta \) with length and bandwidth \( \beta \) times the length and bandwidth of \( g_{n}(m) \). MCPC waveforms can be configured to have zero sidelobes in selected areas in the delay-Doppler plane, based on the belief on target state, in order to improve tracking performance [11].

In this work \( s(m), m = 0, \ldots, M - 1 \) denotes the radar waveform, which can be either a Bjöck CAZAC or an MCPC. Also, let \( \{ \tau, \nu \} \) be the delay-Doppler shift of the received waveform after being reflected from a target, and \( \{ \bar{\tau}, \bar{\nu} \} \) the delay-Doppler shift of one of the template signals in the matched filter (MF) at the receiver. The possible delay-Doppler pairs that appear in the return waveform at time step \( k \) form a finite set of cardinality \( L_{k} \) (see [7] for details). The \( l \)th member of this set corresponds to delay-Doppler pair \( \{ \tau, \nu \} \) while the \( l \)th member corresponds to \( \{ \bar{\tau}, \bar{\nu} \} \). Moreover, a delay-Doppler shifted return is denoted as \( s(l, m) = s(m - \tau)e^{j2\pi m \nu / M_{d}} \) while a template waveform is denoted as \( s(\bar{l}, m) = s(m - \bar{\tau})e^{j2\pi \bar{\nu}m / M_{d}} \), where \( m = 0, \ldots, M_{d} - 1, l = 0, \ldots, L_{k} - 1 \) for both waveforms. \( M_{d} > M \) is chosen to be large enough to accommodate the maximum delay of a signal reflected from a target. Also, let \( S_{k} \) be an \( M_{d} \times L_{k} \) matrix with columns the delayed and Doppler shifted versions of the transmitted waveform \( s(l, m) \).

When the return waveform is sampled at the Nyquist rate and the discrete-time samples are matched filtered with the template waveform, the Nyquist AF results given by [7]:

\[
A_{s}(l, \bar{l}) = \frac{1}{\xi_{s}} \sum_{m=0}^{M_{d}-1} s(l, m)s^{\ast}(\bar{l}, m)
\]  

where \( \xi_{s} = \sum_{m=0}^{M_{d}-1} s(m)s^{\ast}(m) \) is the energy of the transmitted waveform. \( A_{s}(l, \bar{l}) \) also represents the projection of \( s(l, m) \) onto \( S_{k}^{\ast} \), where ‘\( \ast \)’ denotes conjugate transpose.

In Figure 1, the AF surfaces \( |A_{s}(l, \bar{l})|^{2} \) of a Bjöck CAZAC and of an MCPC consisting of Bjöck CAZACs are plotted in subfigures \( a \) and \( b \). The Bjöck CAZAC has (a required prime number) length of 4999. The MCPC has length \( 106 \times 47 = 4992 \) and is composed of \( \beta = 106 \) identical Bjöck CAZACs of prime number length \( \alpha = 47 \). In the MCPC case, the AF surface has non-zero sidelobes for delays \( \tau \) multiples of \( \beta = 106 \). In addition, for \( \tau = 0 \) non-zero sidelobes occur only at \( \nu \) integer multiple of \( \alpha = 47 \). The number of Bjöck CAZACs in the MCPC and their length are adaptively selected in a tracking scenario [11] according to the belief on target state so that \( A_{s}(l, \bar{l}) = 1 \) for \( l = \bar{l} \) and zero otherwise.

3. COMPRESSIVE MEASUREMENT ACQUISITION MATRIX

Two compressive acquisition matrices are considered in this paper. The first, is a matrix with random Gaussian elements [3] denoted as \( \Phi \) which is fixed and agnostic to the received waveform (see [2] for details).

The second, is denoted as \( \Phi_{k} \) and it is configured at each time step \( k \) of the tracking scenario. Both matrices have size \( C \times M_{d} \) projecting the \( M_{d} \) dimensional return waveform onto a \( C < M_{d} \) dimensional space, thus reducing sampling rate and processing complexity. The proposed adaptive construction, however, differs in that it sequentially embeds prior information on possible delay-Doppler shifts in the return waveform. This information is available by a particle filter during tracking [7, 11]. Compressive processing with the adaptive matrix then reduces to a reduction in the compressed AF sidelobes compared to when using a random construction.

The adaptive measurement acquisition matrix \( \Phi_{k} \) is generated at each time step \( k = 1, \ldots, K \) of a tracking scenario. \( \Phi_{k} \) is composed of two matrices, \( Q_{k} \) and \( S_{k} \) and is given by \( \Phi_{k} = Q_{k}S_{k}^{\ast} \). \( S_{k} \) contains prior information, while \( Q_{k} \) provides almost orthogonal compressive measurements.

\( S_{k} \) is an \( M_{d} \times L_{k} \) matrix adaptively constructed every time step \( k \) as mentioned in Section 2. If \( S_{k}^{\ast} \) were used as the acquisition matrix, the Nyquist AF in (2) would result, and \( L_{k} < M_{d} \) measurements would be obtained. However, the goal of CSP is to reduce the number of CS measurements to much less than \( M_{d} \) and \( L_{k} \).

\( Q_{k} \) is a \( C \times L_{k} \) matrix independent of the transmitted waveform. Its columns are \( L_{k} \) delay-Doppler cyclically shifted versions of a Bjöck CAZAC sequence \( q(c), c = 0, \ldots, C - 1 \) of length \( C \), chosen to provide an almost orthogonal set (low AF sidelobes). A Bjöck CAZAC that is cyclically shifted in time by \( \tau \) and Doppler shifted by \( \nu \) is given by \( q(c, l) = q(c - \tau)c^{l}e^{j2\pi \nu l / C}, c = 0, \ldots, C - 1, l = 0, \ldots, L_{k} - 1 \) where the time index is taken modulo \( C \) denoted as \((c - \tau)_{C}\). Moreover, the pair \( \{ \tau, \nu \} \) corresponds to index \( l \). The AF of \( q(c) \) is given by:

\[
A_{q}(l, \bar{l}) = \frac{1}{\xi_{q}} \sum_{c=0}^{C-1} q(c, l)q^{\ast}(c, \bar{l})
\]  

where \( \xi_{q} = \sum_{c=0}^{C-1} |q(c)|^{2} \). From the above \( \Phi_{k} \) has entries

\[
\phi_{k}(c, m) = \sum_{l=0}^{L_{k}-1} q(c, l)s^{\ast}(l, m)
\]  

where \( c = 0, \ldots, C - 1, m = 0, \ldots, M - 1 \).

4. COMPRESSED AMBIGUITY FUNCTIONS

In general, the compressively sensed sequence is given by:
The energy of the compressively sensed waveform becomes the scaled CAZAC in {doppler pairs}

\[ s_{cs}(l, c) = \sum_{m=0}^{M_d-1} \phi(c, m)s(l, m), c = 0, \ldots, C - 1 \]  

Random Matrix: The energy of the compressively sensed waveform is, using the restricted isometry property of the random construction [3, 4], \( \xi_s = \sum_{c=0}^{C-1} s_{cs}(c)s^*_s(c) = \beta \sum_{l,m} s^*_l s_m \) with \( 1 - \beta \leq \beta \leq 1 + \delta \), where \( \delta \in (0, 1) \) is the restricted isometry constant. In order to obtain the compressed AF the CS waveforms are correlated as

\[ A_{cs}(l, l) = \frac{M_d - 1}{\beta C \xi_s} \sum_{c=0}^{C-1} s_{cs}(l, c)s^*_cs(l, c) \]

where \( \frac{M_d}{\beta C \xi_s} \) is the energy normalization factor. As seen above, the resolution in delay-Doppler in the compressed AF is preserved through indices \( l, \bar{l} \) corresponding to delay-doppler pairs \( \{\tau, \nu\} \) and \( \{\bar{\tau}, \bar{\nu}\} \) respectively similarly to the Nyquist AF in (2).

Adaptive Matrix: The compressively sensed waveform is

\[ s_{cs}(l, c) = \sum_{m=0}^{M_d-1} \phi_k(c, m)s(l, m) = \frac{L_k - 1}{\xi_s} \sum_{c=0}^{C-1} q(c, l')A_s(l, l') \]

where (4) and (2) were used. When \( s(l, m) \) is the MCPC, with \( A_s(l, l) = 1 \) and zero sidelobes, the compressively sensed waveform becomes the scaled CAZAC in \( Q_0 \) as \( s_{cs}(l, c) = \xi_s q(c, l) \). For any kind of waveform the compressive AF is

\[ A_{cs}(l, l) = \frac{1}{\xi_s} \sum_{c=0}^{C-1} s_{cs}(l, c)s^*_cs(l, c) \]

which is normalized by \( \xi^2 \xi_q \). With (6) and (7)

\[ A_{cs}(l, l) = \frac{1}{\xi_q} \sum_{c=0}^{C-1} \sum_{l'} q(c, l')A_s(l, l') \sum_{l''} q^*(c, l')A^*_s(l', l''). \]

Specifically for an MCPC waveform \( A_s(l, l) = 1 \) with zero sidelobes the compressed AF reduces to the Nyquist AF of the cyclically shifted Björck CAZAC in matrix \( Q_k \)

\[ A_{cs}(l, l) = A_q(l, l) \]

where (3) was used. The above result justifies the choice of the adaptive acquisition matrix; when using an MCPC waveform together with an adaptive construction the sidelobes of the compressed AF are as low as the sidelobes of the Nyquist AF of a CAZAC in \( Q_k \) of length \( C \). In the random matrix case, however, the sidelobes appear with random levels, which are difficult to adaptively minimize utilizing updated track information. Figures 1c and 1d show typical AF surfaces in the random and adaptive CSP case with \( C = 50, C = 47 \) respectively. If more compressive measurements are taken, then the sidelobes of the compressed AF decrease in both the random and adaptive CSP cases.

For a more complete comparison the averaged peak side-lobe levels (PSL) of the AF in the NSP and the two CSP cases are shown in Figure 2 along with 95% confidence intervals. The PSL [12] represents, in the noiseless case, the most likely erroneous indication on the true target state. Therefore, a low PSL corresponds to improved tracking performance. The PSL was averaged over 100 trials where the random and adaptive matrices were varied. Moreover, the number of compressive measurements taken were \( C = [50, 100, 200, 500] \) and \( C = [47, 101, 199, 499] \) for the random and adaptive CSP respectively. In the adaptive case \( C \) is restricted to be prime as it corresponds to the length of a Björck CAZAC in \( Q_k \). For the Björck CAZAC of the large length \( (M = 4999) \) considered in this paper the sidelobes \( A_s(l', l) \) in (2) for \( l' \neq l \) are very small. For the MCPC the sidelobes are zero and, therefore, \( \mu(\Phi_k, S_k) = \max_{c,l} |q(c, l)| \). Due to the constant amplitude property of the CAZAC, \( |q(c, l)| = \sqrt{1/C} \forall c, l. \) Then \( \mu(\Phi_k, S_k) = \sqrt{1/C}. \) The minimal coherence exhibited by a spike and a Fourier basis [4] is \( \sqrt{1/C} \) while the maximal coherence of \( \Phi_k \) with itself is 1. Therefore, using

5. SPARSITY AND INCOHERENCE

The sparsity requirement of compressive sensing [4] is satisfied for the MCPC and Björck CAZAC waveforms as these are sparse in their respective dictionary \( S_k \) of delay and Doppler shifted versions of the waveform described in Section 2. Sparsity is due to the zero, for the adaptively configured MCPC, or the very low, for the Björck CAZAC, AF sidelobes [7, 11], where the AF represents projections of a waveform onto its associated dictionary \( S_k \).

Next, incoherence is investigated. Coherence is defined as the maximum correlation between any two elements of the acquisition matrix \( \Phi_k \) and the sparsifying dictionary \( S_k \) [4]:

\[ \mu(\Phi_k, S_k) = \max_{c,l} \left| \sum_{m=0}^{M-1} \phi_k(c, m)s(l, m) \right| \]

using (6), normalizing \( q(c) \) and \( s(m) \) so that \( \xi_q = 1 \) and \( \xi_s = 1 \), and using the fact that \( A_s(l, l) = 1 \)

\[ \mu(\Phi_k, S_k) = \max_{c,l} |q(c, l)| + \sum_{l' = 0, l' \neq l}^{J_k - 1} |q(c, l')A_s(l', l)|. \]

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7. REFERENCES


6. CONCLUSIONS

An adaptive compressive sensing acquisition matrix was proposed that produces measurements suitable for compressive processing. Theoretical and numerical results demonstrate that using the adaptive acquisition matrix reduces peak AF sidelobe levels compared to when using a random acquisition matrix. As shown in a companion paper [9] the SNR at the output of the compressed matched filter is also improved when using an adaptive versus a random acquisition matrix.

Fig. 1. NSP and CSP AF surfaces.

Fig. 2. PSL versus the number of compressive measurements for the NSP and the random and adaptive CSP cases. The mainlobe is at 1.

The MCPC as a radar waveform minimizes coherence, while using the Björck CAZAC of large length results to an almost minimum coherence. This shows that the proposed acquisition matrix $\Phi_k$ is appropriate for compressively sensing any type of MCPC or Björck CAZAC radar waveform.

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