SIGNAL RECOVERY IN SHIFT-INVARINT SPACES FROM PARTIAL FREQUENCY DATA

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ABSTRACT

This paper studies conditions under which a signal can be reconstructed from partial frequency content. We focus on signals in shift-invariant spaces generated by multiple generators. For these signals, we derive a lower bound on the necessary signal bandwidth as well as sufficient conditions on the generators such that signal recovery is possible. When the available frequency content is not sufficient to recover the signal, we propose appropriate pre-processing that can improve the reconstruction ability.

Index Terms— Sampling, shift-invariant spaces, sparsity.

1. INTRODUCTION

In many applications we do not have access to the entire signal we wish to process, but only to a filtered version \( y = Fx \) of the signal (cf. Fig.1). In this paper \( F \) will be a linear filter which cuts out some of the frequency content of the signal \( x \). In many practical cases \( F \) is a just a lowpass filter [1]. The goal of this paper is to study under what conditions a signal \( x \) can be recovered from its filtered version \( y = Fx \). Clearly, if we have no prior knowledge on the original signal \( x \), and we are given a filtered signal \( y = Fx \), of which some frequencies are missing, then we cannot recover the missing frequency content. However, if we have prior knowledge on the signal structure then it may be possible to interpolate it from the given data. Our focus here is on signals that lie in shift-invariant (SI) spaces, generated by multiple generators [2].

2. PROBLEM FORMULATION

For \( 1 \leq p \leq \infty \), the common Lebesgue spaces of functions on the real axis \( \mathbb{R} \) and with values in the \( M \)-dimensional Euclidean space \( \mathbb{C}^M \) are denoted by \( L^p(\mathbb{R}, \mathbb{C}^M) \). We write \( L^p \) for \( L^p(\mathbb{R}, \mathbb{C}^1) \), and

\[
\hat{x}(\omega) = \int_{\mathbb{R}} x(t) e^{-i\omega t} dt, \quad \omega \in \mathbb{R}
\]

stands for the Fourier transform of a function \( x \in L^2 \). The conjugate-complex of \( e \in \mathbb{C} \) is denoted by \( \overline{e} \), and \( G^T \) is the transpose of the matrix \( G \). For \( T, \Omega \in \mathbb{R} \), the translation- and modulation operator on \( L^2 \) are defined by \( (Tx)(t) = x(t-T) \) and \( (\Omega x)(t) = x(t)e^{i\Omega t} \), respectively.

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Suppose that a signal \( x \in L^2 \) is filtered by a linear filter with impulse response \( f \) and frequency response \( \hat{f} \)

\[
y(t) = (Fx)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{x}(\omega) \hat{f}(\omega) e^{i\omega t} d\omega.
\]

Here we assume that \( \hat{f} \in L^2 \) and \( \| \hat{f} \|_\infty = \sup_{\omega \in \mathbb{R}} |\hat{f}(\omega)| < \infty \). The later assumption implies that \( F \in L^2 \rightarrow L^2 \) is bounded. The spectral support \( \mathcal{B}(f) := \{ \omega \in \mathbb{R} : \hat{f}(\omega) \neq 0 \ a.e. \} \) of \( F \) is the set of frequencies on which \( \hat{f} \) is non-zero, and its Lebesgue measure \( \lambda[\mathcal{B}(f)] \) is said to be the bandwidth of \( F \). We are going to consider the following two questions

1) What signals \( x \) can be recovered from the output \( y \) of \( Fx \)?
2) Can we perform preprocessing of \( x \) prior to filtering to ensure that \( x \) can be recovered from \( y \)?

If \( x \) is assumed to be in \( L^2 \) and if the bandwidth of \( F \) is finite, then recovery from \( y \) will not be possible in general, since some frequency components of the signal are cut out. However, if \( x \) lies in a certain subspace \( S \) of \( L^2 \) it might still be possible to recover \( x \) from \( y \) by exploiting the structure of \( S \). Here we assume that \( x \) lies in a complex SI subspace with multiple generators. Let \( \phi = \{ \phi^{(1)}, \ldots, \phi^{(N)} \} \) be a given set of \( N \) functions (the \( g \)en\( e\)rators) in \( L^2 \) and \( T \in \mathbb{R} \). Then the SI space defined by \( \phi \) is

\[
S_T(\phi) = \text{span}(\{ T^k \phi^{(n)} : n = 1, \ldots, N ; k \in \mathbb{Z} \})
\]

and every signal \( x \in S_T(\phi) \) has the form

\[
x(t) = \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} a_k^{(n)} \phi_k^{(n)}(t) = \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} a_k^{(n)} \phi_k^{(n)}(t - kT).
\]

In order to guarantee a unique and stable representation of every \( x \in S_T(\phi) \), it is assumed that the sequence \( \{ T^k \phi^{(n)}(\omega) \}_{k \in \mathbb{Z}} \) is a Riesz basis for \( S_T(\phi) \). Then the \( N \) coefficient sequences \( \{ a_k^{(n)} \}_{k \in \mathbb{Z}} \) are in \( \ell^2 \) and these sequences uniquely define the signal \( x \).

With every sequence of the form \( \{ T^k \phi^{(n)}(\omega) \}_{k \in \mathbb{Z}} \), which span an SI space \( S_T(\phi) \), we associate the matrix \( G_\phi(\omega) \) with infinity many rows and \( N \) columns whose entry in the \( k \)th row and \( n \)th column is

\[
[ G_\phi(\omega)]_{k,n} = \tilde{\phi}_k^{(n)}(\omega + k \frac{2\pi}{T}) = (T_k^k \phi^{(n)})(\omega),
\]
and where we defined $\Omega := 2\pi/T$. Moreover, the $N \times N$ matrix
\[
M_\phi(\omega) := \frac{1}{T} \mathbf{G}_\phi(\omega)^T \mathbf{G}_\phi(\omega)
\]
is called the Grammian\(^1\) of the sequence \(\{T_k \phi^{(n)}\}_{k,n}\). It is well known [4] that \(\{T_k \phi^{(n)}\}_{k,n}\) is a Riesz basis for \(\mathcal{S}_T(\phi)\) if and only if there exists only two constants $0 < A_\phi \leq B_\phi < \infty$ such that
\[
A_\phi \mathbf{I}_N \leq M_\phi(\omega) \leq B_\phi \mathbf{I}_N, \quad \text{a.e. } \omega \in [-\Omega/2, \Omega/2].
\]
This condition guarantees that the sequences \(\{a_k^{(n)}\}_{k,n}\) can be recovered from \(x \in \mathcal{S}_T(\phi)\) by means of a linear bounded operator. For the following considerations it will be important that the translation operator \(T_t\), which generates \(\mathcal{S}_T(\phi)\), and the filter operator \(F_f\) commute, i.e. that \(F_f (T_t) = T_t F_f\), which is easily verified.

### 3. RECOVERY CONDITIONS

Let \(x \in \mathcal{S}_T(\phi)\) be arbitrary. Using that \(T_t\) and \(F_f\) commute, the signal \(y = F_f x\) at the output of the filter \(F_f\) can be written as
\[
y(t) = (F_f x)(t) = \sum_{n = 1}^{N} \sum_{k \in \mathbb{Z}} a_k^{(n)} (T_k \hat{\psi}^{(n)})(t) \tag{2}
\]
where \(\hat{\psi}^{(n)} := F_f \tilde{\phi}^{(n)}, n = 1, \ldots, N\) denotes the lowpass filtered generator \(\tilde{\phi}^{(n)}\). Thus the signal \(y\) at the filter output also lies in an SI space generated by the functions \(\psi = \{\psi^{(1)}, \ldots, \psi^{(N)}\}\). Consequently, the coefficients \(\{a_k^{(n)}\}_{k,n}\) can be determined from \(y\) by means of a bounded linear operator if the sequence \(\{T_k \hat{\psi}^{(n)}\}_{k,n}\) forms a Riesz basis for its closed linear span, i.e. if there exists constants $0 < A_\psi \leq B_\psi < \infty$ such that
\[
A_\psi \mathbf{I}_N \leq M_\psi(\omega) \leq B_\psi \mathbf{I}_N, \quad \text{a.e. } \omega \in [-\Omega/2, \Omega/2]. \tag{3}
\]
Since \(\hat{\psi}^{(n)} = \hat{f}(\omega) \hat{\phi}^{(n)}(\omega)\), one easily verifies that the matrix \(\mathbf{G}_\psi(\omega)\) can be written as \(\mathbf{G}_\psi(\omega) = \mathbf{F}(\omega) \mathbf{G}_\phi(\omega)\) where \(\mathbf{F}(\omega)\) is a diagonal matrix with entries
\[
[\mathbf{F}(\omega)]_{k,k} = \hat{f}(\omega + k\Omega) = (T_k \hat{f})(\omega).
\]

Therewith we obtain
\[
M_\psi(\omega) = \frac{1}{T} \mathbf{G}_\phi(\omega)^T \mathbf{G}_\phi(\omega) = \frac{1}{T} \mathbf{G}_\phi(\omega)^T \mathbf{D}(\omega) \mathbf{G}_\phi(\omega)
\]
where \(\mathbf{D}(\omega) = \mathbf{F}(\omega)^T \mathbf{F}(\omega)\) is a diagonal matrix with entries \([\mathbf{D}(\omega)]_{k,k} = \left| \hat{f}(\omega + k\Omega) \right|^2\). In order that the lower bound in (3) is satisfied it is obviously necessary that the matrix \(\mathbf{D}(\omega)\) has rank \(N\) for almost all \(\omega \in [-\Omega/2, \Omega/2]\). This implies immediately a necessary condition on the bandwidth of the filter \(F_f\).

#### Lemma 1

Let \(\{T_k \phi^{(n)}\} : n = 1, \ldots, N ; k \in \mathbb{Z}\) be a Riesz basis for \(\mathcal{S}_T(\phi)\), and let \(\psi^{(n)} = F_f \phi^{(n)}\) for \(n = 1, \ldots, N\). Then a necessary condition for \(\{T_k \psi^{(n)}\} : n = 1, \ldots, N, k \in \mathbb{Z}\) to be a Riesz basis for \(\mathcal{S}_T(\psi)\) is that
\[
\lambda[\mathbf{B}(f)] \geq N \Omega = N \frac{2\pi}{T}.
\]

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\(^1\)It easy to see that \(\{T_k \phi^{(n)}\}_{k,n}\) is an $N$-dimensional stationary sequence with spectral density $\Phi(\theta,T) = M_\phi(-\theta/T)$ [3].

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Lemma 1 gives a lower bound on the bandwidth of the filter $F_f$ such that $x$ can be recovered from $y = F_f x$. It shows that the bandwidth of $F_f$ is lower bounded by the rate of innovation $N/T$ of the signal space. Of course, this necessary condition is generally not sufficient. In order that (3) is satisfied, the filter $F_f$ has to be matched to the generators $\phi$ of the signal space. To explain this, let $\Omega_k := \{w \in \mathbb{R} : k\Omega - \Omega/2 < w < k\Omega + \Omega/2\}$ and denote by $K(f)$ the set of indices $k$ of those frequency intervals $\Omega_k$ on which $f$ is not identical zero, i.e. $K(f) = \{k \in \mathbb{Z} : \lambda[\mathbf{B}(f) \cap \Omega_k] > 0\}$. The cardinality of $K(f)$ is denoted by $K$, and it is clear that $K \geq N$ if $F_f$ satisfies the condition of Lemma 1. Now we define the $K \times N$ matrix $\mathbf{G}_\phi(\omega)$ which consists of those rows of $\mathbf{G}_\phi(\omega)$ which are indexed by $K(f)$. Therewith, we can formulate a sufficient condition such that $x$ can be recovered from $y = F_f x$.

#### Lemma 2

Let $\phi = \{\phi^{(1)}, \ldots, \phi^{(N)}\}$ be the generators of $\mathcal{S}_T(\phi)$, and let $f$ be the impulse response of the filter $F_f$. If there exists two constants $0 < A \leq B < \infty$ such that
\[
A \mathbf{I}_N \leq \mathbf{M}_f(\omega) := \frac{1}{T} \mathbf{G}_\phi(\omega)^T \mathbf{G}_\phi(\omega) \leq B \mathbf{I}_N,
\]
then every $x \in \mathcal{S}_T(\phi)$ can be recovered from $y = F_f x$.

Since the generators $\phi$ are known, it is sufficient to determine the coefficient sequences $\{a_k^{(n)}\}$ to recover $x$. According to (2) and since $\{T_k \hat{\psi}^{(n)}\}_{k,n}$ is required to form a Riesz basis, the coefficients are obtained by $a_k^{(n)} = \langle \tilde{\chi}_k^{(n)} \rangle_{\mathcal{S}_T(\psi)}$ where $\{\tilde{\chi}_k^{(n)}\}_{k,n}$ is the unique dual basis of $\{T_k \hat{\psi}^{(n)}\}_{k,n}$. If $\hat{\chi}^{(n)}(\omega) = [\hat{\psi}^{(1)}(\omega), \ldots, \hat{\psi}^{(N)}(\omega)]^T$ denotes the $N$-dimensional vector containing the Fourier transforms of the generators $\psi$, then the dual basis is generated by the functions with Fourier transform
\[
\tilde{\chi}(\omega) = [\tilde{\chi}^{(1)}(\omega), \ldots, \tilde{\chi}^{(N)}(\omega)]^T = [\mathbf{M}_\psi(\omega)]^{-1} \hat{\chi}(\omega).
\]

The overall recovery scheme is sketched in Fig. 2. The observed signal $y$ is pre-filtered by $N$ linear filters with frequency response $\tilde{\chi}^{(n)}$. The outputs are uniformly sampled at rate $1/T$. This gives the $N$ coefficient sequences $\{a_k^{(n)}\}$. These sequences are modulated by $T$-periodic impulse trains and filtered by analog filters $\phi^{(n)}$ whose impulse responses are the generators of our signal space. We refer to [1] for a more detailed and slightly different description.

### 4. PREPROCESSING

Suppose we are given the generators $\phi$ of the signal space $\mathcal{S}_T(\phi)$ and the filter $F_f$. Assume that $F_f$ fulfills the condition of Lemma 1 but that Lemma 2 is not satisfied, i.e. a signal $x \in \mathcal{S}_T(\phi)$ can generally not be recovered from the filtered signal $y = F_f x$. In this situation it might be possible to pre-process $x$ prior to filtering $w = F_f x$ to ensure that $x$ can be recovered from the filtered $y = F_f w$. (cf. Fig. 3).
Single Channel We confine ourselves to bounded linear preprocessing operators \( P \) on \( L^2 \) and we assume (as for the filter \( F_f \)) that \( P \) commutes with the translation operator \( T_2 \) which generates the signal space \( S_T(\phi) \), i.e. we assume that \( PT_2 = T_2P \). If \( P \) and \( F_f \) commute, then preprocessing will not improve the ability to recover \( x \). This follows from the fact that in this case, Fig. 3 is equivalent to first filtering the signal \( w = P_f x \) followed by post-processing \( y = P w \). However, we assume that \( x \) cannot be recovered from \( w = P_f x \), which means that part of the signal \( x \) disappeared in the kernel of the operator \( F_f \). Consequently there is no linear operator \( P \) which can recover \( x \) from \( P w = P F_f x \). This observation excludes e.g. linear filters of the form (1) as preprocessors [1].

As an alternative, we consider a preprocessor of the form

\[
(P_p x)(t) = p(t) x(t)
\]

where \( p \in L^\infty \) is a \( T \)-periodic bounded function on \( \mathbb{R} \). The periodicity of \( p \) ensures that \( P_p T_2 = T_2 P_p \) whereas the boundedness implies the boundedness of \( P_p : L^2 \rightarrow L^2 \). Since \( p \) is \( T \)-periodic, it can be written as \( p(t) = \sum_{k \in \mathbb{Z}} b_k e^{i2\pi kt/T} \) with the sequence of Fourier coefficients \( \{b_k\}_{k \in \mathbb{Z}} \subseteq \ell^2 \). For every signal \( x \in S_T(\phi) \), the output of the preprocessor is then

\[
w(t) = (P_p x)(t) = \sum_{n=1}^{N} a_n^{(n)} (T^n_2 \xi^{(n)})(t)
\]

where

\[
\xi^{(n)}(t) := (P_p \phi^{(n)})(t) = p(t) \phi^{(n)}(t)
\]

\[
v = \sum_{k} b_k \phi^{(n)}(t) e^{i2\pi kt/T} = \sum_{k} b_k (M_2^{(n)} \phi^{(n)})(t).
\]

Thus the preprocessed signal \( w = P_p x \) lies in the SI space \( S_T(\xi) \) spanned by the functions \( \xi = \{\xi^{(1)}, \ldots, \xi^{(N)}\} \) given by (5). Moreover, according to (6), each \( \xi^{(n)} \) lies in a modulation-invariant space

\[
M_2^{(n)}(\phi^{(n)}) = \text{span} \{M_2^{(n)} \phi^{(n)} : k \in \mathbb{Z}\}.
\]

Evidently, in the preprocessing simply changes the generators of the signal space from \( \phi \) to \( \xi \). The entries of the matrix \( G_\xi(\omega) \), associated with the new SI signal space \( S_T(\xi) \), are given by

\[
(G_\xi(\omega))_{k,n} = \xi^{(n)}(\omega + k\Omega) = \sum_{l \in \mathbb{Z}} b_l \phi^{(n)}(\omega + [k + l]\Omega) .
\]

This shows that \( G_\xi(\omega) = B G_\phi(\omega) \) where \( B \) is a double infinite matrix containing the Fourier coefficients of \( p \):

\[
B_{n,m} = b_{m-n}.
\]

Now the preprocessed signal \( w = P_p x \) is filtered by \( F_f \). As in Section 3, this yields the signal \( y = F_f w \) of the form (2) where the filtered generators \( \psi^{(n)} \) are now given by \( \psi^{(n)}(t) = F_f \xi^{(n)} = F_f P_p \phi^{(n)} \). Consequently \( x \) can be recovered from \( y \) if (3) is satisfied but where the matrix \( M_\psi(\omega) \) is now given by

\[
M_\psi(\omega) = \frac{i}{2} G_\phi(\omega)^T B^T D(\omega) B G_\phi(\omega) .
\]

Multiple Channels We next consider the case where we have \( M \) parallel preprocessing channels, as shown in Fig. 4. In each channel the signal is preprocessed using a mixer of the form (4) but with a different \( T \)-periodic mixing function

\[
p_m(t) = \sum_{k \in \mathbb{Z}} b_k^{(m)} e^{i2\pi kt/T} , \quad m = 1, \ldots, M
\]

in each channel. The preprocessed signals are then filtered by \( F_f \) which is assumed to be equal in each channel. The analysis is similar to the single channel case: For every \( x \in S_T(\phi) \), let \( w_m(t) = (P_{p_m} x)(t) \) be the signal at the output of the preprocessor in the \( m \)th channel and set \( w(f) = [w_1(t), \ldots, w_M(t)]^T \). Then

\[
w(t) = \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} a_k^{(n)} (T^n_2 \xi^{(n)})(t)
\]

where \( \xi^{(n)}(t) := [\xi^{(n)}_1(t), \ldots, \xi^{(n)}_M(t)]^T \in L^2(\mathbb{R}, \mathbb{C}^M) \) and where the individual entries are defined similar as in (5), i.e.

\[
\xi^{(n)}(t) := (P_{p_m} \phi^{(n)})(t) = \sum_{k} b_k^{(m)} (M_2^{(n)} \phi^{(n)})(t).
\]

Equation (9) shows that the preprocessed signal \( w(t) \) lies in an SI subspace of \( L^2(\mathbb{R}, \mathbb{C}^M) \) spanned by the sequence \( \xi = \{T^n_2 \xi^{(n)}\}_{k,n} \). It is not hard to see that the Grammian, associated with this sequence, is equal to the sum of the Grammians of the individual sequences \( \xi_m = \{T^n_2 \xi^{(n)}\}_{k,n} \) in each channel, i.e.

\[
M_\xi = \sum_{m=1}^{M} M_\xi_m = \frac{1}{2} \sum_{m=1}^{M} G_\phi(\omega)^T B^T_m B_m G_\phi(\omega) .
\]

where the double infinite matrix \( B_m \) contains the Fourier coefficient of the \( m \)th mixing function \( p_m \) as in (7).

Finally the signal \( w_m(t) \) in each channel is filtered by \( F_f \). This yields, similar to the single channel case, the output signal

\[
y(t) = [y_1(t), \ldots, y_M(t)]^T = \sum_{n=1}^{N} \sum_{k \in \mathbb{Z}} a_k^{(n)} (T^n_2 \psi^{(n)})(t)
\]
where $\psi^{(n)}(t) = [\psi_1^{(n)}(t), \ldots, \psi_M^{(n)}(t)]^T$ with entries $\psi_m^{(n)}(t) = (Ff\xi^{(n)}_m)(t)$. This shows that also the output signal $y$ lies in an $\mathcal{S}$ space spanned by the sequence $\psi = \{T^k\psi^{(n)}(k)\}_{k,n}$. Consequently, the signal $x$ (i.e., the coefficient sequences $\{a_k^{(n)}\}_{k\in \mathbb{Z}}$) can be recovered from $y$ by means of a bounded linear operator if the Grammian $M_\phi(\omega)$ of the sequence $\{T^k\psi^{(n)}(k)\}_{k,n}$ satisfies (3). Combining the consideration which gave (8) and (10), one obtains for the Grammian $M_\phi(\omega)$ in the multi channel case the expression

$$
M_\phi(\omega) = \frac{1}{T} G_\phi(\omega)^T \left( \sum_{m=1}^{M} B_m^T D(\omega) B_m \right) G_\phi(\omega). \tag{11}\n$$

In order that $M_\phi(\omega)$ can satisfy the lower bound in (3), it is necessary that the double infinite matrix in the brackets of (11) has full rank $N$ for almost all $\omega \in [-\Omega/2, \Omega/2)$. Since this matrix is now the sum of $M$ matrices, the necessary condition on the filter bandwidth is lowered by a factor of $1/M$ compared to Lemma 1.

**Lemma 3:** Let $\{T^k\phi^{(n)}(k) : n = 1, \ldots, N ; k \in \mathbb{Z}\}$ be a Riesz basis for $\mathcal{S}_T(\phi)$ and let $\psi^{(n)} = Ff\phi^{(n)}$. For $n = 1, \ldots, N$ set $\psi^{(n)} := [\psi_1^{(n)}(t), \ldots, \psi_M^{(n)}(t)]^T$. Then a necessary condition for $\{T^k\psi^{(n)}(k) : n = 1, \ldots, N ; k \in \mathbb{Z}\}$ to be a Riesz basis for $\mathcal{S}_T(\psi)$ is that

$$
\lambda[B(f)] \geq \frac{N}{\mathcal{T}_L} \Omega = \frac{N}{\mathcal{T}_L} 2\frac{\omega}{\mathcal{F}}. \n$$

Consequently, increasing the number $M$ of preprocessing channels decreases the necessary bandwidth of the filters $F_f$ in every channel. Conversely, given the bandwidth of the filter $F_f$, Lemma 3 gives a minimal necessary number of preprocessing channels. Using more preprocessing channels than the minimal necessary will usually increase the frame bound $A_\phi$ in (3) and thus will increase stability of the signal recovery. Moreover, the matrices $B_m$, $m = 1, \ldots, M$, containing the Fourier coefficients of the mixing functions, can be used to match the generators $\phi$ to the filter $F_f$ such that Lemma 2 is satisfied. Recovery works exactly as described in Section 3, where the calculation of the dual basis is slightly different.

**5. DISCUSSION**

**Example** To illustrate our approach we consider the example where $F_f$ is an ideal lowpass with bandwidth $\Omega$, i.e., $\hat{f}(\omega) = 1$ for all $\omega \in [-\Omega/2, \Omega/2)$ and $\hat{f}(\omega) = 0$ for all other frequencies. The signal space $\mathcal{S}_T(\phi)$ is assumed to be spanned by $N = 3$ generators $\phi^{(n)}$, $n = 1, 2, 3$. We consider the $m$th channel of the preprocessing bank in Fig. 4. Let $G_{\psi^{(n)}}(\omega) = F(\omega) B_m G_{\phi}(\omega)$ be the matrix associated with the sequence $\{T^k\psi^{(m)}(k)\}_{k,n}$. By the assumption on $F_f$, the diagonal matrix $\phi(\omega)$ has only one non-zero entry. Consequently, $G_{\psi^{(n)}}(\omega)$ is essentially a $1 \times 3$ matrix (all other rows are identically zero) with entries

$$
[G_{\psi^{(n)}}(\omega)]_{0,n} = \sum_{n \in \mathbb{Z}} b_{1,n}^{(m)} \hat{\phi}^{(n)}(\omega + \Omega), \quad n = 1, 2, 3. \tag{12}\n$$

According to (11) the overall Grammian is given by

$$
M_\phi(\omega) = \frac{1}{T} \sum_{m=1}^{M} G_{\psi^{(n)}}(\omega)^T G_{\psi^{(n)}}(\omega). \n$$

As you can see, $M_\phi(\omega)$ is the sum of $M$ rank one matrices, so that we need at least $M = 3$ channels for (3) to be satisfied. Moreover, the sequences $\{b_{k,n}^{(m)}\}_{k \in \mathbb{Z}}$ have to be chosen such that the $1 \times 3$ matrices $G_{\psi^{(n)}}(\omega)$ are linearly independent for every $\omega \in [-\Omega/2, \Omega/2]$. In particular, if the generators $\phi^{(n)}(\omega)$ are piecewise constant on every interval $\Omega_n$ then the choice of the sequences $\{b_{k,n}^{(m)}\}_{k \in \mathbb{Z}}$ is a simple linear algebra problem. Equation (12) shows that having a coefficient $b_{k,n}^{(m)} \neq 0$ means that the frequency content of $\phi^{(n)}(\omega)$ in the interval $\Omega_n$ is shifted to the frequency band $\Omega_n$ and can therefore pass the filter $F_f$ in the $m$th channel. This example was used, e.g. in [5, 6], for blind reconstruction of multiband signals.

**Sparse Signals** Finally we sketch how a sparsity model of the signal $\{\tilde{L}^n\}$ can be incorporated into the above approach. To this end, it is assumed that only $L$ out of $N$ generators $\phi^{(n)}$ are active, i.e. that at most $L$ of the coefficient sequences $\{a_k^{(n)}\}_{k \in \mathbb{Z}}$ are nonzero. Generally, $N$ is much larger than $L$ and we do not know which of the generators are active. The sparsity assumption implies that the sequences $\{a_k^{(n)}\}$ in the recovery scheme of Fig. 2 share a joint sparsity pattern with at most $L$ out of $N$ non-zero sequences. This in turn allows to recover $x$ from only $2L$ sampling banks in Fig. 2. In this case, the $N$ filters $\tilde{\chi}^{(n)}$ can be replaced by $2L$ filters $\tilde{\varphi}^{(n)}$ which are obtained by an appropriate linear combination (cf. (7)) of the filters $\tilde{\chi}^{(n)}$. This yields only $2L$ sequences $\{\tilde{\alpha}^{(n)}_{k,l}\}_{k \in \mathbb{Z}}$. Following the separation idea in [8], it is possible to determine the $L$ active generators from these 2L sequences by solving a finite dimensional optimization problem. Recovery is then obtained by applying results regarding infinite measurement models to our problem [7, 8]. The reduction in the number of sampling filters corresponds to a reduction in the necessary bandwidth of the known signal. To see this, note that if only $L$ generators are active, then the sequence $\{T^k \phi^{(n)}(k)\}_{k \in \mathbb{Z}}$ is a frame (and not a Riesz basis) for the actual signal space which is spanned by the $L$ generators. Consequently, the Grammian $M_\phi(\omega)$ (and in turn $M_\phi(\omega)$) need to have only rank $L$ for every $\omega \in [\Omega/2, \Omega/2)$ in order that the coefficient sequences can be recovered from $x$ and $y$, respectively. However, since we do not know which generators are active, the Grammians need to have at least rank $2L$ following standard arguments of compressed sensing. The concrete example considered in [5] illustrates this behavior.

**6. REFERENCES**


