COMPACT SUPPORT KERNELS BASED TIME-FREQUENCY DISTRIBUTIONS: PERFORMANCE EVALUATION

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ABSTRACT

This paper presents two new time-frequency distributions based on kernels with compact support (KCS) namely the separable CB (SCB) and the polynomial CB (PCB) TFDs. The implementation of these distributions follows the method developed for the Cheriet-Belouchrani CB TFD. The performance of this family of TFDs is compared to the most known quadratic distributions through tests on multi-component signals with linear and nonlinear frequency modulations (FMs) considering the noise effects as well. Comparisons are based on the evaluation of an objective criterion namely the Boashash-Sucic’s normalized instantaneous resolution performance measure that allows to provide the optimized TFD using a specific methodology. In all presented examples, the KCS TFDs have been shown to have a significant interference mitigation, with the component energy concentration around their respective instantaneous frequency laws being well preserved giving high resolution measure values.

Index Terms— Time-frequency distribution, compact support kernel, separable compact support kernel, polynomial compact support kernel, performance evaluation

1. INTRODUCTION

Time-frequency distributions (TFDs) are the natural choice that allows to analyze and process nonstationary signals accurately and efficiently by performing a mapping of one-dimensional signal \( x(t) \) into a two dimensional function of time and frequency \( TFD_x(t,f) \). Herein, we are interested by the quadratic class of TFDs, also known in the literature as kernel-based transform [1]:

\[
TFD_x(t,f) = \int_\infty^{-\infty} \int_\infty^{-\infty} e^{j2\pi(s-t)} \phi(\eta,\tau) x(s+\tau/2) x^*(s-\tau/2) e^{-j2\pi f \eta} \eta d\eta ds d\tau
\]  

where \( \phi(\eta,\tau) \) is a two dimensional kernel. Unlike the Gaussian kernel that suffers from information loss due to reduction in accuracy when the Gaussian is cut off to compute the time-frequency distribution [2], kernels with compact support (KCS), derived from the Gaussian kernel, are found to recover this information loss, improves processing time and retains the most important properties of the Gaussian kernel [3]. These features are achieved thanks to the compact support analytical property of this type of kernels since they vanish themselves outside a given compact set. It turns out that through a control parameter of the kernel width, the corresponding time-frequency distributions allow a better elimination of cross-terms while providing good resolution in both time and frequency. Motivated by these interesting characteristics, we propose in this contribution the use of two new kernels with compact support derived from the Gaussian kernel for time-frequency analysis namely the separable KCS (SKCS) [4] and the polynomial KCS (PKCS) [5]. Similarly to the CB TFD [6], the induced TFDs are referred to as SCB TFD and PCB TFD, respectively. Then, comparisons are established between results obtained by using the KCS based TFDs and the most commonly used time-frequency representations. In order to provide objective assessment, our comparison is based on the Boashash-Sucic resolution performance measure [7]:

\[
P_i = 1 - \frac{1}{3} \frac{A_{m}}{A_{m}} + \frac{A_{s}}{2A_{m}} + (1 - S)
\]

where \( A_{m} \), \( A_{s} \), \( A_{x} \) are respectively the average amplitudes of the main-lobes, side-lobes and cross-terms of two consecutive signal components, with \( S = (B_1 + B_2)/[2(f_2 - f_1)] \) being a measure of the components’ separation in frequency \( B_k \) and \( f_k \), \( k = 1, 2 \), are respectively the instantaneous bandwidth and the instantaneous frequency (IF) of the \( k^{th} \) component. \( P_i \) is close to 1 for well-performing TFDs and 0 for poorly-performing ones. An overall measure \( P \) is taken to be the median of the instantaneous measures \( P_i \) corresponding to different time slices in the relevant sections of the signals. This quantitative criterion permits the performances’ evaluation of different distributions and can be used for adaptive and automatic parameters selection in \( t-f \) analysis.
2. THE SEPARABLE KCS BASED TFD

In [4], a separable kernel family with compact support (SKCS) applied in image processing was introduced. The latter is a separable version of the compact support kernel [3]. Hence, the CB kernel also referred to as KCS can be modified to the separable form that we will call Separable Cheriet-Belouchrani (SCB) kernel yielding to a new time-frequency distribution of quadratic class referred to as SCB TFD. Recall that the CB kernel is defined as [6]

\[
\phi_{CB}(\eta, \tau) = \begin{cases} 
\frac{C}{A e^{(\eta^2 + \tau^2)/D^2 - 1}} & \text{if } \frac{\eta^2 + \tau^2}{D^2} < 1 \\
0 & \text{Otherwise}
\end{cases}
\]

where \(D \) and \( A = e^C \) are control parameters. The derived SCB kernel is given by

\[
\phi_{SCB}(\eta, \tau) = \begin{cases} 
\phi_{CB}(\eta, 0)\phi_{CB}(0, \tau) & \text{if } \begin{cases} \eta^2 < D^2 \\
\tau^2 < D^2 \end{cases} \\
0 & \text{Otherwise}
\end{cases}
\]

Thus

\[
\phi_{SCB}(\eta, \tau) = \begin{cases} 
\frac{CD^2}{A e^{\eta^2 - D^2 + \tau^2 - D^2}} & \text{if } \begin{cases} \eta^2 < D^2 \\
\tau^2 < D^2 \end{cases} \\
0 & \text{Otherwise}
\end{cases}
\]

The separable CB (SCB) TFD is thus expressed as

\[
SCB_x(t, f) = \int_{-D}^{D} \int_{-\infty}^{+\infty} J_{SCB}(s - t, \tau)x(s + \tau/2) x^*(s - \tau/2)e^{-j2\pi f\tau}dsd\tau
\]

where

\[
J_{SCB}(s', \tau) = A^2 e^{CD^2/(\tau^2-D^2)} \int_{-D}^{D} e^{CD^2/(\eta^2-D^2)} e^{j2\pi \eta s'} d\eta
\]

with \(s' = s - t\). Note that, as for the CB kernel, the resulting SCB TFD is always real-valued, conserves energy since \(\phi_{SCB}(0, 0) = e^0 = 1\) and verifies translation covariance with respect to time and frequency.

3. THE POLYNOMIAL KCS BASED TFD

In order to avoid the Gaussian kernel’s drawbacks, two approaches exist [5]: approximating the Gaussian kernel by a finite support kernel, or defining new kernels with properties close to the Gaussian. In [5], a new compact support kernel of polynomial form, referred to as PKCS, was proposed and applied to scale-space image processing. This compact support nature together with the possibility of controlling the kernel’s window width lead us to propose a new kernel for time-frequency analysis called Polynomial Cheriet-Belouchrani (PCB) kernel yielding to a new quadratic time-frequency distribution referred to as PCB TFD. The latter is implemented following the same procedure as for the CB TFD and the SCB TFD. The PCB kernel is defined as

\[
\phi_{PCB}(\eta, \tau) = \begin{cases} 
\frac{\gamma^1}{\pi\lambda^{\gamma+1}} \left(\lambda^2 - (\eta^2 + \tau^2)\right)^{\gamma} & \text{if } (\eta^2 + \tau^2) < \lambda^2 \\
0 & \text{Otherwise}
\end{cases}
\]

where \(\lambda\) is the radius of the kernel support and \(\gamma\) is considered to be a positive integer so that the resulting kernel has a polynomial form [5]. The polynomial CB (PCB) TFD is thus formulated as

\[
PCB_x(t, f) = \int \int J_{PCB}(s - t, \tau)x(s + \tau/2) x^*(s - \tau/2)e^{-j2\pi f\tau}dsd\tau
\]

where

\[
J_{PCB}(s', \tau) = \frac{\gamma + 1}{\pi\lambda^{\gamma+1}} \int \frac{\sqrt{\lambda^2 - \eta^2}}{\lambda^2 - \lambda^2 - \eta^2} \left(\lambda^2 - (\eta^2 + \tau^2)\right)^{\gamma} e^{j2\pi \eta s'} d\eta
\]

The PCB TFD is real-valued and verifies translation covariance with respect to time and frequency as all time-frequency distributions of the quadratic class. Note that the kernel width is controlled through the parameter \(D\) for both the CB TFD and the SCB TFD and \(\lambda\) for the PCB TFD; and its peak is adjusted through the parameter \(A = e^C\) for the CB and SCB TFDs and \(\gamma\) for the PCB TFD permitting a tradeoff between a good auto-term resolution and a sufficient cross-term suppression.

4. RESULTS AND COMMENTS

4.1. Example 1: Noisy closely spaced chirps

Herein, we consider a multi-component signal \(s_1(t)\) of length \(N = 128\) that consists of two closely spaced parallel linear FMs with frequencies increasing from 0.15 to 0.25 Hz and from 0.2 to 0.3 Hz, respectively; embedded in additive white Gaussian noise, with signal-to-noise ratio of 10 dB. The time-frequency plots of the optimized TFDs under the constraints of Boashash-Sucic’s criterion are shown in Fig. 1. From visual inspection, it can be seen that the KCS based TFDs perform much better than the other considered TFDs since they generate the most appealing plots. Table 1 records the numerical results of the optimization procedure over the entire time interval [1,128] and reveals that the optimal TFD of the noisy signal \(s_1(t)\) is the CB TFD with smoothing parameters \(D = 2.5\) and \(A = 0.11\) since it possesses the largest value of the overall resolution performance measure \(P\).
Table 1. Optimization results of example 1 (robustness to noise test).

<table>
<thead>
<tr>
<th>TFD</th>
<th>Optimal kernel Parameters</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>N/A</td>
<td>0.6442</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>Bartlet, $L = 71$</td>
<td>0.8222</td>
</tr>
<tr>
<td>BJD</td>
<td>N/A</td>
<td>0.6764</td>
</tr>
<tr>
<td>CWD</td>
<td>$\sigma = 0.9$</td>
<td>0.7173</td>
</tr>
<tr>
<td>ZAMD</td>
<td>$\alpha = 0.56$</td>
<td>0.6422</td>
</tr>
<tr>
<td>CB TFD</td>
<td>$D = 2.5, A = 0.11$</td>
<td>0.8443</td>
</tr>
<tr>
<td>SCB TFD</td>
<td>$D = 3, A = 0.28$</td>
<td>0.8439</td>
</tr>
<tr>
<td>PCB TFD</td>
<td>$\lambda = 2, \gamma = 1$</td>
<td>0.8363</td>
</tr>
</tbody>
</table>

4.2. Exemple 2: Mixture of linear and nonlinear FMs

Here, we consider a synthetic signal $s_2(t)$ consisting of two intersecting sinusoidal FMs and two non-parallel, non-intersecting chirps. The nonlinear components consist of an increasing and decreasing sinusoidal frequency modulated signals at $t_0 = 0, f(t_0) = 0.35$ Hz, having both a period $T = 128$ sec with smallest and highest frequencies equal to 0.25 Hz and 0.45 Hz, respectively. The two chirps occupy the frequency ranges $f = 0.16 - 0.19$ Hz and $f = 0.07 - 0.1$ Hz, respectively. The smallest frequency separation between the linear and nonlinear components is between 0.18 – 0.25 Hz near 97 sec and it is low enough and is just avoiding intersection. This example confirms the effectiveness of the KCS based kernels in detecting closely spaced components. Fig. 2 shows the superiority of the KCS based kernels and the spectrogram over the other quadratic time-frequency distributions in resolving the four closely spaced components as well as in reducing the cross-terms. Herein, the Boashash-Sucic’s procedure is applied twice in order to measure the parameters for each of the pairs of consecutive components with equal amplitudes of each TFD time slice. The optimizing TFD’s parameters are chosen so that they produce the greatest value of the Boashash-Sucic’s overall performance measure for both the two linear chirps ($P^{(1)}$) and the two sinusoidal FMs ($P^{(2)}$); the resulting $P$ to maximize is equal to $(P^{(1)} + P^{(2)})/2$. Table 2 presents the numerical results of the optimization procedure and indicates that the CB TFD with parameters $A = 1.2$ and $D = 3$ is the optimal TFD for representing $s_2(t)$ since it produces the largest value of $P$.

Table 2. Optimization results of example 2.

<table>
<thead>
<tr>
<th>TFD</th>
<th>Optimal kernel Parameters</th>
<th>$P^{(1)}$</th>
<th>$P^{(2)}$</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>N/A</td>
<td>0.6428</td>
<td>0.6089</td>
<td>0.6258</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>Hanning, $L = 45$</td>
<td>0.8741</td>
<td>0.8644</td>
<td>0.8692</td>
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<tr>
<td>BJD</td>
<td>N/A</td>
<td>0.6699</td>
<td>0.7623</td>
<td>0.7246</td>
</tr>
<tr>
<td>CWD</td>
<td>$\sigma = 0.45$</td>
<td>0.7602</td>
<td>0.7687</td>
<td>0.7644</td>
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<tr>
<td>ZAMD</td>
<td>$\alpha = 0.9$</td>
<td>0.7352</td>
<td>0.7381</td>
<td>0.7366</td>
</tr>
<tr>
<td>CB TFD</td>
<td>$D = 2.2, A = 3$</td>
<td>0.8780</td>
<td>0.8786</td>
<td>0.8783</td>
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<tr>
<td>SCB TFD</td>
<td>$D = 3, A = 4$</td>
<td>0.8701</td>
<td>0.8802</td>
<td>0.8751</td>
</tr>
<tr>
<td>PCB TFD</td>
<td>$\lambda = 0.2, \gamma = 4$</td>
<td>0.8813</td>
<td>0.8701</td>
<td>0.8757</td>
</tr>
</tbody>
</table>

5. CONCLUSION

Assuming both linear and nonlinear FM laws, very closely spaced multi-component signals and noise effects, it is shown that the KCS based TFDs outperform other TFDs in suppressing the cross-terms, while still achieving the best time-frequency resolution. The conducted comparisons are quantified using the Boashash-Sucic’s objective criteria. KCS based TFDs give in all studied cases the largest performance measure values compared to the most known and powerful time-frequency representations. In addition, they reveal the best information in terms of detection of the component number, extraction of the IF laws, estimation of signal component bandwidths and evaluation of side-lobe and cross-term amplitudes. The latter are the best eliminated using KCS kernels thanks to their compact support nature and the flexibility in tuning the kernel parameters in order to reach their optimization. A real time DSP implementation of the investigated distributions will constitute the subject of our future publications.

6. REFERENCES

Fig. 1. Optimized TFDs over the full duration $T = 128$ of the signal of example 1 composed of two parallel LFM s with frequency ranges spreading from 0.15 to 0.25 Hz and 0.2 to 0.3 Hz, respectively; embedded in 10 dB AWGN. (a) WVD, (b) Spectrogram (Bartlett, $L = 71$), (c) BJD, (d) CWD ($\sigma = 0.9$), (e) ZAMD ($\alpha = 0.56$), (f) CB TFD ($D = 2.5$, $A = 0.11$), (g) SCB TFD ($D = 3$, $A = 0.28$) and (h) PCB TFD ($\lambda = 2$, $\gamma = 1$).

Fig. 2. Optimized TFDs over the time duration $[1, 128]$ of the signal $s_2(t)$ composed of two non-parallel, non-intersecting chirps and two intersecting sinusoidal FMs. (a) WVD, (b) Spectrogram (Hanning, $L = 45$), (c) BJD, (d) CWD ($\sigma = 0.45$), (e) ZAMD ($\alpha = 0.5$), (f) CB TFD ($D = 1.2$, $A = 3$), (g) SCB TFD ($D = 3$, $A = 4$) and (h) PCB TFD ($\lambda = 0.2$, $\gamma = 4$).