TLS-FOCUSS FOR SPARSE RECOVERY WITH PERTURBED DICTIONARY

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ABSTRACT

In this paper, we propose a new algorithm, TLS-FOCUSS, for sparse recovery for large underdetermined linear systems, based on total least square (TLS) method and FOCUSS (FOCal Underdetermined System Solver). The problem of sparse recovery when perturbations appear in both the measurements and the dictionary (sensing matrix) is considered. FOCUSS algorithm is extended with main idea of TLS to reduce the impact of the perturbation of dictionary on the performance of sparse recovery. The simulation results illustrate the advantage of TLS-FOCUSS on accuracy and stability compared with ordinary FOCUSS algorithm.

Index Terms—TLS-FOCUSS, underdetermined system, sparse solution, maximum a posteriori (MAP) estimate

1. INTRODUCTION

Because of its widespread application, sparse recovery has been the hot spot of research since the middle of this decade. Various algorithms for solving the underdetermined linear equations have been proposed and can be roughly divided into two classes: convex optimization [1] and greedy algorithms [2]. It is well known that the methods based on convex optimization has low convergence rate so that its computational burden is high. On the other hand, the performance of greedy algorithm can only be guaranteed when the sensing matrix (Dictionary) satisfy some rigorous conditions, such as very small restricted isometry constants [3].

FOCUSS was originally designed to obtain a sparse solution by successively solving quadratic optimization problems [4][5] and was widely used to deal with compressed sensing problems[6]. Compared with other approaches, the obvious advantage of FOCUSS is its fast convergence. For FOCUSS, only a few iterations tends to be enough to obtain a rather good approximating solution. Meanwhile, its requirement on sensing matrix is much looser than that of greedy methods.

FOCUSS is a excellent choice of algorithm of sparse recovery for large scale application.

Recently, much attention had been paid on the perturbation of observation data and sensing matrix in sparse linear regression model. Original FOCUSS [4] only allowed noise item to be added to sample data. Corruption of sensing matrix (e.g., model error, which is the main error source of solution), has not been considered. Harman [7] analyzed the effect of perturbation of sensing matrix on the performance of BP (Basis Pursuit) algorithms and found the condition of stable recovery for BP; Zhu [8] adopted TLS (Total Least Square) technique to treat the perturbation and a new framework based on Lasso was proposed. But the discussion was still limited on BP and convex optimization methods. As the effective solver of large scale sparse recovery problem, FOCUSS is lacking any extension suitable for perturbed model.

The main contribution of this paper is an extension of FOCUSS to a novel compressed sensing method using Total Least Square procedure. Our objective is to overcome effectively the influence of perturbation of sensing matrix on the accuracy of sparse recovery. Meanwhile, the merit of FOCUSS, rapid convergence and good adaption to intrinsic properties of sensing matrix, is maintained.

2. PERTURBED LINEAR REGRESSION MODEL

According to [7], the perturbed linear regression model can be formulated as follows:

\[ y = (A + E)x + e \]  

where \( x \in \mathbb{R}^n \) is unknown vector, sensing matrix \( A \in \mathbb{R}^{m \times n} \) is perturbed by unknown matrix \( E \) with the same dimension, and \( y \in \mathbb{R}^m \) is observation vector which is corrupted by noise \( e \in \mathbb{R}^m \). (1) can be rewritten as

\[ (B + D)z' = 0 \]  

where

\[ B = [-y, A], \quad D = [e, E], \quad z' = [1, x^T]^T \]  

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It is assumed that the components of perturbation matrix $D$ are independent and identically Gaussian distributed. In most compressed sensing application, $x$ is supposed to be sparse, that is, the number of its nonzero component is tiny. Furthermore, (1) is heavily underdetermined. It means $m \ll n$.

3. TLS-FOCUSS ALGORITHM

Now we will propose a extension of FOCUSS, TLS-FOCUSS, to solve (1) using Bayesian framework [5] and main idea of TLS.

3.1. Bayesian Formulation

From Bayesian viewpoint, unknown vector $x$ is supposed to be random and independent of $D$. Then the maximum posterior (MAP) estimation of $x$ can be written as:

$$\hat{x}_{\text{MAP}} = \arg \max_x \ln p(x|y)$$

$$= \arg \max_x [\ln p(y|x) + \ln p(x)]$$

From (1), we obtain

$$y - Ax = G(x)v$$

where $G(x) = [1, x^T] \otimes I$, $v = \text{vec}(D)$, here vec() is matrix vectorizing operator and $\otimes$ represents Kronecker product. Noticing that $v \sim N(0, \sigma^2 I)$ and $G(x)G^H(x) = (1 + \|x\|^2)I$, we have

$$\ln p(y|x) = \frac{1}{2\sigma^2} \left( \frac{(y - Ax)^H(y - Ax)}{1 + \|x\|^2} + C_1 \right)$$

Just as [5], the distribution of sparse vector $x$ could be Generalized Gaussian Distribution,

$$p(x) = C_2 \exp \left( -\frac{1}{2\beta p} \sum_{k=1}^{m} |x[k]|^p \right)$$

where $C_2$ is constant, $p \in (0, 1)$ is index of $l_p$ norm and $\beta$ is constant depend on $p$.

From (1), (7) and (8), we have

$$\hat{x}_{\text{MAP}} = \arg \min_x \left[ \frac{\|y - Ax\|^2}{1 + \|x\|^2} + \gamma \|x\|^p \right]$$

and $\gamma = \sigma^2 / \beta p$.

3.2. Derivation of TLS-FOCUSS

The optimization problem (9) is equivalent to

$$\min_{z'} \left[ \frac{\|Bz'\|^2}{1 + \|z'\|^2} + \gamma \|z'\|^p \right]$$

where $z', B$ are defined as (3). With normalization on $z'$, we have

$$\min_z [\|Bz\|^2 + \gamma \|z\|^p], \quad \text{s.t} \|z\|^2 = 1$$

Using Lagrange multiplier method, the objective function can be written as

$$J(z) = \|Bz\|^2 + \gamma \|z\|^p + \lambda (1 - z^H z)$$

then the optimal solution $z^*$ must satisfy $\frac{\partial J}{\partial z} = 0$, then

$$(B^H B + \alpha \Pi(z^*))z^* = \lambda z^*$$

where

$$\alpha = p\gamma / 2, \quad \Pi(z) = \text{diag}\left( \left[ |z(i)|^{p-2} \right]_{i=1, \ldots, n+1} \right)$$

It is easily seen that $\lambda$ should be the minimal eigenvalue of matrix $B^H B + \alpha \Pi(z_{k-1})$. But it’s very hard to find it for two reasons: Firstly, the minimal eigenvalue is likely close to zero for the matrix above is approximately singular; Secondly, the dimension of matrix above is tremendous for most large scale application, which lead to forbidden computational burden for matrix inversion. However, (15) implies that

$$(B^H B + \alpha \Pi(z_{k-1}))^{-1} z_k = \frac{1}{\lambda} z_k$$

By (16), finding the minimal eigenvalue is changed to finding the maximal eigenvalue. The latter become much more well-posed. Moreover, with the aid of matrix inversion formula, we have

$$(B^H B + \alpha \Pi(z_k))^{-1} = \frac{1}{\alpha} (W(z_k) - W(z_k)B^H (\alpha I - BW(z_k)B^H)^{-1} BW(z_k))$$

where $W(z_k) = \Pi(z_k)$. Let

$$Q(z_k) = (W(z_k) - W(z_k)B^H (\alpha I - BW(z_k)B^H)^{-1} BW(z_k))$$

we obtain

$$Q(z_{k-1}) z_k = \frac{\alpha}{\lambda} z_k$$

It should be mentioned that the dimension of matrix $\alpha I - BW(z_k)B^H$ is much less than that of matrix $B^H B + \alpha \Pi(z_k)$, so the cost of matrix inversion is extremely reduced; On the other hand, we need only calculate the maximal eigenvalue and corresponding eigenvector instead of all the eigenvalue and eigenvector of $\alpha I - BW(z_k)B^H$. That is to say, some highly efficient solver, such as Lanczos iteration, could be utilized to make the problem further simplified.

3953
Noting that the optimal problem (9) is non-convex, the TLS-FOCUSS algorithm guarantees convergence to a local optimum. Once the initial point $z_0$ is close to the true point, estimation of true value can be found through iterations. In this paper, we set $x_0 = A^H(\alpha A^H)^{-1} y$, then $z_0$ is set through substituting $x_0$ into (3) and normalization.

Following is the algorithmic description of TLS-FOCUSS.

**Algorithm 1 (TLS-FOCUSS)**

**Input:** $z_0$, $B$, $\alpha$, $p$.

1. Set $W_k = \text{diag}(\|z_{k-1}(i)\|^2 )_{i=1,\ldots,n+1}$, and $p \in (0, 1)$.
2. Compute $Q_k = W_k - W_k B^H (\alpha I + BW_kB^H)^{-1}BW_k$.
3. Compute the largest eigenvalue $\lambda_k$ and corresponding eigenvector $u_k$ with Lanczos method, and Set $z_k = u_k$.
4. If $\|z_k - z_{k-1}\|^2 / \|z_{k-1}\|^2 < \epsilon$, exit; else goto step 1.

### 4. SIMULATION RESULTS

This section shows the advantages of recovering ability of TLS-FOCUSS from noisy measurements and perturbed dictionary with numerical simulation. Let $x$ be a $s$-sparse vector, i.e. $\|x\|_0 = s$, and let the average power of $x_T$ be normalized. The $m \times n$ dictionary $A$ is chosen as Gaussian random matrix, entries of $A$ is independently Gaussian distributed with mean zero and variance 1, Entries of matrix $D$ in (3), including perturbation on measurement and dictionary, are also independently Gaussian distributed with mean zero and variance $\sigma^2$. Then overall SNR can be represented as $1/\sigma^2$. $p$ is set to 0.5.

#### 4.1. Simulation 1: ability of distinguishing weak signal

The ability of algorithms to extract weak signals is shown. Choose a strong signal, a weak signal and a middle ones for non-zero vector $x_T$ as follows: $x_T = [0.4139, -0.9186, -1.4819]^T$. SNR is set to 20dB, the number of rows and columns of dictionary matrix are 36 and 512, respectively.

It can be seen from Fig.1 that TLS-FOCUSS does much better than FOCUSS in extracting weak signal when dictionary and measurement are both corrupted. For TLS-FOCUSS, the position and amplitude of strong and weak signal are both recovered excellently. However, the result of FOCUSS for weak signal is buried in "False Peak" brought by perturbation on dictionary and can not be distinguished correctly.

Statistical results are showed in Fig.2. Every experiment is carried out independently for 1000 times with different SNR. Fig.2(A) shows that the successful probability of TLS-FOCUSS is higher than that of FOCUSS. Fig.2(B) shows root-mean-square error (RMSE) of signal amplitude recovery of FOCUSS and TLS-FOCUSS. And we can find that RMSE curves of TLS-FOCUSS algorithm are much lower.

### 4.2. Simulation 2: recovery of signals with the same amplitude

In this simulation we focus on recovery of signals with the same amplitude. We set $s = 3, x_T = (1, -1, 1)^T$, $m = 128, n = 512$, and $T$ is a random subset of $[1, 2, \cdots, n]$. RMSE of recovered amplitudes in Fig.3 shows that the recovery performance of TLS-FOCUSS in this scenario is much better than that of FOCUSS and its extension, regularized FOCUSS.

### 5. CONCLUSION

In this paper, we propose a new algorithm, TLS-FOCUSS, to recover the sparse vector from an underdetermined sys-
tem when the measurements and dictionary matrix are both perturbed. The simulations show the our approach performs much better than the original FOCUSS and its regularized version, which do not consider the dictionary perturbation, in behaviors of probability of success in finding sparse support and RMSE of signal amplitude recovery when the dictionary of model is perturbed. The benefic of TLS-FOCUSS make it be a good candidate of sparse recovery algorithms for more practical applications.

6. REFERENCES