ADJUGATE PAIRS OF SPARSE ARRAYS FOR SAMPLING TWO DIMENSIONAL SIGNALS

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ABSTRACT

Sparse sampling with coprime lattice arrays was introduced recently in the literature. It has been shown that a dense coarray can be constructed from such a pair of arrays, and is useful in array processing and image processing applications. For example, the coarray allows one to identify many more sources than sensors. After a brief review of these fundamentals, this paper examines the case where the two arrays are generated by matrices that are adjugates of each other. In this case it is possible to obtain a dense rectangular tiling of the 2D frequency plane from a pair of coarse 2D DFT filter banks. The special case where the adjugate pairs are generated by skew circulant matrices has some advantages, which we shall elaborate.

Index Terms— Sparse sensing, adjugates, lattice arrays, multidimensional arrays, coarrays, DFT filter banks.

1. INTRODUCTION

The idea of sparse sampling with a pair of coprime lattice arrays was introduced in [10], [11]. A sampling matrix in D dimensions is a \( D \times D \) nonsingular matrix \( M \) with sample locations at the points \( Mn \) where \( n \) is an integer vector. This set of points, called the sampling array, is the lattice generated by \( M \), denoted as \( LAT(M) \) [2], [8]. In [10] we considered two sampling arrays generated by integer matrices \( M \) and \( N \) (Fig. 1(a), (b)). The sampling densities of these arrays are \( 1/\det M \) and \( 1/\det N \) respectively. In general the integers \( \det M \) and \( \det N \) are considerably greater than unity, hence the sampling arrays are sparse with respect to unity. As explained in [10] the importance of such sampling arrays arises from considering the difference coarray [4] of this pair of arrays, defined to be the set of all integer vectors of the form

\[
k = Mn_1 - Nn_2
\]

where \( n_i \) are integer vectors. The density of points on this coarray can be made equal to unity by constraining the integer matrices \( M \) and \( N \) to be commuting and coprime.\(^1\) So, even though the individual arrays are sparse nonrectangular lattices, the coarray can be forced to be a dense rectangular lattice as demonstrated in Fig. 1(c).\(^2\)

To appreciate the importance of this, consider a multidimensional wide sense stationary (WSS) signal \( x(t) \). The correlation between the outputs of two elements (one from each array) is \( R(k) = E[x(Mn_1)x^*(Nn_2)] = E[x(n)x^*(n - k)] \) where \( k = Mn_1 - Nn_2 \). Thus the autocorrelation samples \( R(k) \) can be estimated at any multidimensional lag given by the coarray element (1). This has applications in multidimensional spectrum computation and DOA estimation. For example, one can use two coprime arrays with a total of \( \det M + \det N \) elements to identify \( O(\det MN) \) sources. Similarly it is possible to combine two 2D-DFT filter banks, one with \( \det M \) subbands and the other with \( \det N \) subbands, to create as many as \( \det MN \) subbands, obtaining a dense tiling of the 2D frequency plane \([0, 2\pi)^2\). Details can be found in [10].

In this paper we elaborate on the case where \( N \) is the adju- gate of \( M \). In this case the matrices automatically commute. An important advantage is that it is possible to obtain a dense rectangular tiling of the frequency plane starting from the two sparse coprime arrays. For the specific case where the matrices are skew-circulants, the corresponding lattice offers certain advantages, which we shall elaborate.

![Fig. 1](image-url)

\(^1\)Two multidimensional sampling lattices are called left coprime if the generating integer matrices \( M \) and \( N \) are left coprime. This means that if an integer matrix \( L \) is a left common factor i.e., \( M = LM_1 \) and \( N = LN_1 \), then \( L \) is unimodular (\( \det L = \pm 1 \)). Right coprimality is defined similarly.

\(^2\)Commuting coprime matrices have had an important role in the theory of multidimensional multirate systems [1], [3].

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2. TILING THE FREQUENCY PLANE WITH COPRIME ARRAYS

We first review from [10] how two 2D nonseparable DFT filter banks on the individual sparse sampling lattices (an N-DFT filter bank and a M-DFT filter bank, one in conjunction with each lattice array), can be combined in such a way that a much higher resolution is achieved in the 2D frequency plane. Consider the D-dimensional array of sensors located on the lattice \( LAT(N) \), with \( h(n) \) denoting the tap multiplier in location \( n \). The array response is

\[
H(N^T \Omega) = \sum_n h(n)e^{-j(N^T \Omega)^T n} \tag{2}
\]

Assume \( H(\Omega) = \sum_n h(n)e^{-j\Omega^T n} \) is lowpass\(^3\) with passband \( FPD(2\pi M^{-T}) \) (or \( SPD(\pi M^{-T}) \) if it should be centered at zero).\(^4\) Then \( H(N^T \Omega) \) has the lowpass band \( FPD(2\pi (NM)^{-T}) \), and furthermore, it has extra copies of this passband, located at the points of the scaled reciprocal lattice \( LAT(2\pi N^{-T}) \). From this array we can create a filter bank with shifted array responses as follows:

\[
H_k(N^T \Omega) = \sum_n \left( h(n)e^{j2\pi k^T M^{-1} n} \right) e^{-j(N^T \Omega)^T n} = H(N^T (\Omega - 2\pi N^{-T} M^{-T} k))
\]

where \( k \in FPD(M^T) \). There are \( |\det M| \) such filters. They have responses \( H(N^T \Omega) \) shifted to the scaled reciprocal lattice points \( 2\pi LAT((NM)^{-T}) \). Next, starting from the array with sensors on \( LAT(M) \), we create a filter bank with filters

\[
G_j(M^T \Omega) = G\left(M^T (\Omega - 2\pi M^{-T} N^{-T} j)\right), \tag{3}
\]

\( j \in FPD(N^T) \). There are \( |\det N| \) such filters. If \( G(\Omega) \) has passband \( FPD(2\pi N^{-T}) \), then \( G(M^T \Omega) \) has passband \( FPD(2\pi (MN)^{-T}) \) (together with its shifted copies on the scaled reciprocal lattice \( LAT(2\pi M^{-T}) \)). Thus, when \( MN = NM \), the passbands of \( H(N^T \Omega) \) and \( G(M^T \Omega) \), centered at zero frequency, are identical. Notice how commutativity of \( M \) and \( N \) come into picture. Assuming that \( H(\Omega) \) and \( G(\Omega) \) are lowpass, the passbands of the filters \( H_k(N^T \Omega) \) have distinct center frequencies

\[
2\pi N^{-T} n + 2\pi N^{-T} M^{-T} k, \tag{4}
\]

\( n \in FPD(N^T), k \in FPD(M^T) \). Similarly the passbands of \( G_j(M^T \Omega) \) have distinct center frequencies

\[
2\pi M^{-T} m + 2\pi M^{-T} N^{-T} j, \tag{5}
\]

\( m \in FPD(M^T), j \in FPD(N^T) \). The following two claims are proved in [11].

1. Let \( i \) be an integer in \( FPD((MN)^T) \). Then there exists a pair of filters \( H_k(N^T \Omega) \) and \( G_j(M^T \Omega) \) such that they both have passbands centered at \( 2\pi (MN)^{-T} i \). The product

\[
F_{jk}(\Omega) = G_j(M^T \Omega) H_k(N^T \Omega) \tag{6}
\]

therefore has a passband centered at \( 2\pi (MN)^{-T} i \).

2. Given a pair of filters \( H_k(N^T \Omega) \) and \( G_j(M^T \Omega) \), there is only one overlap of passbands. That is, out of the \( |\det N| \) passbands of \( H_k(N^T \Omega) \) and \( |\det M| \) passbands of \( G_j(M^T \Omega) \), only one pair of passbands has identical center frequencies.

Thus, the \( |\det MN| \) product-filters \( F_{jk}(\Omega) \) have \( |\det MN| \) distinct passbands. Each product filter has precisely one passband, and these together cover the entire frequency range \([0, 2\pi)^D \). The importance of the product (6) is that it is actually possible to realize all the \( |\det MN| \) filters defined by this product by using one of two methods described in [9] for the one dimensional case, namely the passive method (which requires a statistical averaging of snapshots), or the active method (as in radar systems). Consider the example

\[
M = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}. \tag{7}
\]

With \( H(\Omega) \) designed to be lowpass with passband region \( 2\pi FPD(M^{-T}) \) (recentered as shown), a DFT filter bank based on this would have filters with the various passbands shown in Fig. 2(a). Similarly from \( G(\Omega) \) we get the DFT filter bank shown in Fig. 2(b). On the other hand if we build a DFT filter bank starting from \( H(N^T \Omega) \) and \( G(M^T \Omega) \) and combine them as described above, we get a filter bank \( F_{jk}(\Omega) \) with passbands (Fig. 2(c)) as narrow as if we had a filter bank with \( |\det MN| \) channels!

3. TILING WITH ADJUGATE PAIRS OF ARRAYS

In this section we consider the case where the lattice generator \( N \) is the so-called adjugate of \( M \). The importance of this will be clear as the discussion progresses. Recall first that the inverse of \( M \) can be written as

\[
M^{-1} = \frac{\tilde{M}}{\det M} \tag{8}
\]

where \( \tilde{M} \) is a matrix whose elements are the cofactors of elements in \( M \). The matrix \( \tilde{M} \) is called the adjugate of \( M \) [5]. Note that \( M \tilde{M} = (\det M) I \). If we choose \( N = \tilde{M} \), then

\[
MN = NM = c I \tag{9}
\]

where \( c = \det M \). Thus the matrices automatically commute, and the product is proportional to \( I \). In the \( 2 \times 2 \) case, \( M \) and its adjugate can be written as

\[
M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix} \tag{10}
\]
We know that the FPD of the matrix described in Sec. 2. If the matrices are adjugate pairs, then (Ω obtain a dense rectangular tiling of the 2D frequency plane rather than an arbitrary lattice tiling. Thus we are able to plane is proportional to 

In this case \( \hat{M} \) is also the adjugate of \( \hat{M} \), so we say that \( M \) and \( \hat{M} \) form an adjugate pair. Notice that \( \det M = \det \hat{M} = ps - qr \), \hspace{1cm} (11) 

and \( M \hat{M} = \hat{M} M = (ps - qr)I \). \hspace{1cm} (12) 

We know that the FPD of the matrix \((MN)^{-T}\) can be used to tile the \((\Omega_1, \Omega_2)\) plane using a pair of DFT filter banks, as described in Sec. 2. If the matrices are adjugate pairs, then since \( MN = cI \), this tiling becomes a rectangular tiling, rather than an arbitrary lattice tiling. Thus we are able to obtain a dense rectangular tiling of the 2D frequency plane \((\Omega_1, \Omega_2)\) using sparse (non-rectangular) coprime lattices, as demonstrated in Fig. 3. \hspace{1cm} 5 

The density of tiling in the \((\Omega_1, \Omega_2)\) plane is proportional to \( \det MN = (ps - qr)^2 \) even though the densities of the two sparse arrays in space are proportional to \( 1/\det M = 1/(ps - qr) \). This is a unique combination of densities that arises from the use of adjugate pairs to generate the 2D arrays \( LAT(M) \) and \( LAT(N) \). The following result is proved in [12]:

**Theorem 1.** The adjugate pair \((10)\) is coprime if and only if the determinant and trace of \( M \) are coprime, that is, \( (ps - qr, p + s) = 1 \) \hspace{1cm} (13) 

where \((a, b)\) denotes the gcd of the integers \( a \) and \( b \). \hspace{1cm} ⊗

\hspace{1cm} 5For the case where the arrays are used in beamforming for narrowband spatial waves, we have \( \Omega_1 = \pi \sin \phi \cos \theta \) and \( \Omega_2 = \pi \sin \phi \sin \theta \) where \( \theta \) and \( \phi \) are the azimuth and elevation angles. In this case the circular disk in Fig. 3 determines the visible region. Though this disk cannot be tiled perfectly by rectangles (as the rectangles near the circular edge are chopped), the tiling is nearly perfect if the rectangles are small enough. Notice also that since the correspondence between \((\Omega_1, \Omega_2)\) and the azimuth-elevation variables \((\theta, \phi)\) is nonlinear, the tiling in the \((\theta, \phi)\) is nonuniform. So each rectangle in the \((\Omega_1, \Omega_2)\) plane corresponds to a unique, though nonrectangular, region of the \((\theta, \phi)\) plane.

4. SKEW CIRCULANT PAIRS OF LATTICE ARRAYS

In this section we use a skew circulant \( M \) and its adjugate:

\[
M = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}
\] \hspace{1cm} (14)

It is shown in [7] that these are coprime if and only if \( (p + q, p - q) = 1 \). \hspace{1cm} (15)

So (13) is equivalent to (15) in this case. This condition simply says that the two DFT coefficients of the sequence \( \{p, q\} \) are coprime. It is also shown in [7] that this is equivalent to the following conditions: (a) \( (p, q) = 1 \), and (b) \( p \) and \( q \) have opposite parity (one even, the other odd). Lattices generated by skew circulants have interesting properties. Observe that \( M^T M = cI \) \hspace{1cm} (16)

where \( c = p^2 + q^2 \), so the matrix is orthogonal. So the two vectors defining the two sides of \( FPD(M) \) are mutually orthogonal. Furthermore, the generator of the reciprocal lattice is \( \hat{M}^{-T} = c^{-1} M \), which equals \( M \) except for scale. So the reciprocal lattice looks exactly like the original lattice except that it is scaled by a constant \( 1/c \). As an example, consider

\[
M = \begin{bmatrix} n+1 & n \\ -n & n+1 \end{bmatrix}
\] \hspace{1cm} (17)

which has \( \det M = 2n^2 + 2n + 1 \). Fig. 4 shows \( FPD(M) \) and \( FPD(M^{-T}) \). Note that for large \( n \) these look almost like symmetric diamonds. Since \( n \) and \( n + 1 \) are coprime with opposite parity, this matrix and its adjugate:

\[
M = \begin{bmatrix} n+1 & n \\ -n & n+1 \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} n+1 & -n \\ n & n+1 \end{bmatrix}
\] \hspace{1cm} (18)

make a commuting coprime pair.

**Numerical example.** The following example was presented in [10], but without the above theoretical background on adjugates and skew-circulants. It is repeated here for completeness. Let

\[
M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}
\] \hspace{1cm} (19)
Then \( MN = NM = 5I \), so the matrices commute. Furthermore, \( M \) and \( N \) are coprime because Eq. (15) is satisfied. In this case \( \det M = \det N = 5 \), and \( \det MN = 25 \). Thus the two filter banks \( \{ H_k(N^T \Omega) \} \) and \( \{ G_j(M^T \Omega) \} \) have 5 filters each, whereas there are 25 distinct filters in Eq. (6). Thus, with the help of the two 5-point DFT filter banks, we can tile the frequency domain with as many as 25 tiles! The dense tiling of the frequency plane is rectangular (because \( MN \) is proportional to identity), even though the sparse lattices in the spatial domain are nonrectangular. A simple frequency-sampling technique was adopted to design the filters, with \( 64 \times 64 \) samples in \( [-\pi, \pi] \). The resulting \( 64 \times 64 \) rectangular IDFT was taken to be the impulse response \( h(n) \) in Eq. (2), and the filter bank \( H_k(N^T \Omega) \) was constructed. The filter bank \( G_j(M^T \Omega) \) was similarly constructed. Figure 5 shows contour plots for the lowpass filters \( H(\Omega) \) and \( G(\Omega) \) (top). Notice the passband regions \( SPD(\pi M^{-T}) \) and \( SPD(\pi N^{-T}) \). The bottom left of Figure 5 shows the contour plots for the 25 filter products (6). The bottom right shows the simulated version with 800 snapshots, and shows how beautifully the 25 rectangular tilings can be realized in practice. In the simulation, single frequency inputs (impinging waves) were used to generate responses at the outputs of the filters \( H_k(N^T \Omega) \) and \( G_j(M^T \Omega) \), and these responses were multiplied and averaged over 800 snapshots. This was repeated for a dense set of frequencies, and for all the 25 combinations of \( k \) and \( j \), to obtain the 25 contour plots.

5. CONCLUDING REMARKS

In this paper we focused on special classes of DFT filter banks derived from coprime pairs of arrays and pointed out their advantages. Adjugate pairs of arrays were addressed, and it was shown that a dense rectangular tiling of the frequency plane was possible by starting from a sparse pair of coprime arrays. The special case of skew circumulant adjugate pairs was considered in detail. Topics for further study include the development of 2D direction finding algorithms in the coarray domain similar to what has been developed in one dimension [6]. More specifically, since the number of freedoms in the coprime coarray is much larger than that in the original sparse arrays, it should be possible to exploit this to identify many more sources than the number of physical array elements as in [6]. It appears that a significant amount of interesting future research is possible in these directions.

6. REFERENCES