AN EFFICIENT PEAK FREQUENCY ESTIMATOR FOR PRODUCT HIGH-ORDER AMBIGUITY FUNCTION

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ABSTRACT

In this paper, we consider improvement of parameter estimation accuracy of multicomponent polynomial-phase signals (mc-PPSs) using the product high-order ambiguity function (PHAF). We propose a fine search method that improves the coarse estimate obtained as the position of maximum PFAF bin. The method is iterative and based on the dichotomous search. It significantly reduces the calculation complexity compared to the oversampling method. The PFAF-based estimates are, however, biased and the bias can be removed using computationally intensive nonlinear least squares (NLS) methods. These methods require initial estimates that are usually obtained from the PFAF. If the initial estimates are provided by the proposed method, the execution time can be significantly reduced.

Index Terms—Product high-order ambiguity function, parameter estimation, Fourier transform, dichotomous search.

1. INTRODUCTION

The most common model used in parametric analysis of non-stationary signals is a polynomial-phase signals (PPSs) model. The interest in PPSs is motivated by a number of applications including radar, sonar, biomedicine, seismology, geophysics. Numerous methods for the PPS estimation have been proposed [1–4]. The most popular approach is based on the high-order ambiguity function (HAF) [1, 2]. The HAF is a nonlinear operator and its performance is limited when multicomponent PPSs (mc-PPSs) are considered, which is due to the interactions among the components, referred to as cross-terms. Cross-terms give rise to undesired peaks in the HAF spectrum, which can exceed the desired ones. In addition, the HAF suffers from the identifiability problem when some components have the same highest order phase coefficients.

The product high-order ambiguity function (PHAF) introduced in [3] resolves the identifiability problem and effectively suppresses the noise. However, the PFAF-based estimation is biased and the error-propagation makes the estimation of lower order phase coefficients and amplitudes less accurate. The parameter estimation can be improved using an nonlinear least squares (NLS) approach [4, 5]. In [4], the authors proposed an NLS approach that uses the Nelder-Mead simplex algorithm (NMSA) for minimizing the nonlinear cost function. Parameter estimations provided by the PFAF initialize the algorithm. The drawback to this method is heavy computational burden.

In this paper, we propose an efficient algorithm for the mc-PPS parameter estimation from the PFAF. With calculation of only several additional PFAF points, we can estimate the peak frequency very accurately, thus preventing the need for oversampling. The method is based on the dichotomous search of the PFAF peak [6,7]. The estimation bias can be removed by using the NMSA. The execution time of the NMSA can be significantly reduced if we initialize it with the estimates output by the proposed method.

Paper is organized as follows. Section 2 covers the theoretical background regarding the PFAF. The proposed fine search method is presented in Section 3. Simulations are given in Section 4, and conclusions are drawn in Section 5.

2. PRODUCT HIGH-ORDER AMBIGUITY FUNCTION

The multilag high-order instantaneous moment (ml-HIM) of a signal $x(n), n = 0, \cdots, N - 1,$ is defined as [3]

$$
\begin{align*}
    x_1(n) &= x(n), \\
    x_2(n; \tau_1) &= x_1(n + \tau_1) x_1^*(n - \tau_1), \\
    x_3(n; \tau_2) &= x_2(n + \tau_2; \tau_1) x_2^*(n - \tau_2; \tau_1), \\
    &\vdots \\
    x_p(n; \tau_{p-1}) &= x_{p-1}(n + \tau_{p-1}; \tau_{p-2}) x_{p-1}^*(n - \tau_{p-1}; \tau_{p-2}),
\end{align*}
$$

(1)

where $\tau_i = [\tau_1, \tau_2, \cdots, \tau_i], i = 1, ..., P - 1,$ are the sets of used time lags. The multilag HAF (ml-HAF) is defined as the discrete Fourier transform (DFT) of the ml-HIM [3],

$$
X_p(f; \tau_{p-1}) = \sum_{n=0}^{N-1} x_p(n; \tau_{p-1}) e^{-j2\pi fn}.
$$

(2)
When the considered signal \( x(n) \) is a monocomponent \( P \)th order PPS, i.e.
\[
x(n) = A e^{j2\pi \sum_{m=0}^{P} \alpha_m(n\Delta)^m},
\]
where \( \alpha_m \) are polynomial coefficients and \( \Delta \) is the sampling interval, the \( P \)th order ml-HIM of \( x(n) \) is a complex sinusoid with frequency [3]
\[
f = 2^{P-1}\Delta^P P!\alpha_P \prod_{k=1}^{P-1} \tau_k.
\]
The coefficient \( \alpha_P \) can be therefore estimated by searching for the position of maximum in the ml-HAF, and, in turn, the obtained estimate \( \hat{\alpha}_P \) can be used to reduce the phase order by dechirping \( x(n) \) as
\[
x_1(n) = x(n) e^{-j2\pi \hat{\alpha}_P(n\Delta)^P}.
\]
The coefficient \( \alpha_{P-1} \) can be now estimated by searching for the position of maximum in the \((P-1)\)th order ml-HAF of \( x_1(n) \). In this way, all lower order coefficients can be estimated [1, Section III].

When \( x(n) \) is a mc-PPS, i.e.
\[
x(n) = \sum_{k=1}^{K} A_k e^{j2\pi \sum_{m=0}^{P} \alpha_{k,m}(n\Delta)^m},
\]
where \( \alpha_{k,m} \) are coefficients of the \( k \)th component, the \( P \)th order ml-HIM will contain \( K \) sinusoids that correspond to autoterms, each having the frequency proportional to the highest order phase coefficient, according to (4). In addition to the autoterms, the ml-HIM will contain a large number of cross-terms which are, in general, \( P \)th-order PSSs. Specially, when the highest order phase coefficients of some components coincide, the corresponding cross-terms are complex sinusoids, implying that some of the peaks in the ml-HAF correspond to the cross-terms [3]. The maxima-based estimation of phase coefficients is ambiguous, since a peak corresponding to a cross-term can lead to false estimation.

The effect of cross-terms can be significantly reduced using the PHAF proposed in [3]. In the PHAF, \( Q \) sets of time lags are used,
\[
T_{P-1}^Q = \left[ \tau_{P-1}^{(1)}, \tau_{P-1}^{(2)}, \cdots, \tau_{P-1}^{(Q)} \right],
\]
where \( \tau_{P-1}^{(q)} = \left[ \tau_1^{(q)}, \tau_2^{(q)}, \cdots, \tau_{P-1}^{(q)} \right] \) and \( q = 1, \ldots, Q \). The PHAF is defined as
\[
X_P^Q(f, T_{P-1}^Q) = \prod_{q=1}^{Q} X_P(\beta(q)f, \tau_{P-1}^{(q)}),
\]
with the scaling coefficient
\[
\beta(q) = \frac{\prod_{k=1}^{P-1} \tau_k^{(q)}}{\prod_{k=1}^{P-1} \tau_k^{(1)}}.
\]
The PHAF uses the fact that the frequencies of autoterms in the ml-HIM are proportional to the product of used time lags. On the other hand, the frequencies of cross-terms are not proportional to this product. Therefore, the autoterms are enhanced more significantly than the cross-terms, since they are at the same positions in the scaled frequency for each ml-HAF, which is not the case for the cross-terms.

The optimal lags for the \( P \)th order ml-HAF are all equal to each other and to [3]
\[
\tau_{opt} = \frac{N}{2P}.
\]

Assuming that the number of components \( K \) is known [2, 3], we can estimate the phase coefficients of each component starting from the strongest one, as proposed in [8]. This procedure can be summarized by the following three steps.

Step 1. Set \( k = 1 \).

Step 2. Estimate the phase coefficients \( \alpha_{p,p} = 1, 2, \cdots, P \), of the strongest component and filter it out by excising the DC component of the dechirped signal
\[
x_1(n) = x(n) e^{-j2\pi \sum_{p=0}^{P} \hat{\alpha}_p(n\Delta)^p}.
\]

Step 3. Set \( k = k + 1 \). If \( k > K \) exit; otherwise, go to Step 2.

How can we perform the frequency estimation in the PHAF domain? The frequency estimation provided by the position of PHAF maximum bin is suboptimal and further improvement is necessary. The straightforward improvement of the frequency estimation would be to zero-pad the ml-HIMs prior to the calculation of the ml-HAFs and PHAF. This approach, however, suffers from high computational complexity. A better approach would be to perform the oversampling only around the maximum PHAF bin. In the following section, however, we propose a fine frequency estimator that further reduces the number of required operations.

### 3. DICHOTOMOUS SEARCH OF THE PHAF PEAK

In this section, we present a fine search procedure based on the dichotomous search [6]. It is a binary search method where the DFT peak is located and the frequency estimation is adjusted toward the larger of two DFT coefficients from either side of the located peak. New DFT coefficient is calculated halfway between the peak and the larger coefficient. The position of the calculated coefficient represents improved frequency estimation over the initial one. This procedure is iterated \( L \) times. In order to approach the Cramér-Rao lower bound (CRLB), the method, however, requires the zero-padding of the data to at least 1.5 times the original length. This drawback has been overcome by an algorithm modification proposed in [7]. We will herein use a similar approach that also does not require the zero-padding.

The proposed method can be described as follows:
Step 1. Calculate the PHAF $X_Q(f, T_{p-1})$ according to (8) and find the position $f_m$ of its maximum. Denote $I_0 = X_Q(f_m, T_{p-1})$. Calculate the PHAF at two points $f_m \pm \Delta f/2$, i.e.

$$I_{\pm 1} = X_Q(f_m \pm \Delta f/2, T_{p-1}),$$

where $\Delta f$ is the frequency resolution.

Step 2. Iterate $L$ times

$$\Delta f = \Delta f/2$$

if $I_1 > I_{-1}$ then $I_{-1} = I_0$ and $f_m = f_m + \Delta f$
else $I_1 = I_0$ and $f_m = f_m - \Delta f$
calculate $X_Q(f_m, T_{p-1})$ and set $I_0 = X_Q(f_m, T_{p-1})$.

Step 3. The final frequency estimation is obtained as

$$\hat{f} = f_m.$$

Numerous other fine search algorithms are derived for the Gaussian noise environment, which is not the case here. Of particular importance is an iterative technique proposed in [9] that is optimal for the Gaussian noise environment. With the dichotomous search, however, the only assumption regarding the peak shape is that it is a monotonically increasing function in the interval $[f_l - \Delta f/2, f_l]$ and monotonically decreasing in $[f_l, f_l + \Delta f/2]$, where $f_l$ is the true signal frequency.

The proposed method requires calculation of additional $L + 2$ PHAF points. Division by 2 and comparisons can be neglected compared to the PHAF calculation. The complexity of the PHAF calculation is described in [3]. On the other hand, interpolation around the PHAF maximum bin with $N_i$ points entails the calculation of $N_i$ PHAF points. In order to achieve the same performance as the proposed method, the interpolation has to be done with $N_i$ that is much bigger than $L$, as it is shown in the Simulations section.

4. SIMULATIONS

The performance of the proposed method is evaluated for a two-component PPS signal embedded in white Gaussian noise with zero mean and variance $\sigma^2$. Both components are third-order PPSs with phase coefficients given in Table 1. Without loss of generality, we adopted the zero initial phase for both components. Components I and II are characterized by the signal-to-noise ratio (SNR) of 15dB and 13dB, respectively. The SNR is defined by $SNR = 10 \log_{10}(A_k/\sigma)$, where $A_k$ is the amplitude of the $k$th component, $k = 1, 2$. In addition, the signal length is $N = 1024$ and $\Delta = 1/N$. The third-order PHAF is calculated using five sets of time lags, $\tau_2^{(1)} = (\frac{17}{129}, \frac{29}{129}), \tau_2^{(2)} = (\frac{1}{16}, \frac{14}{16}), \tau_2^{(3)} = (\frac{17N}{129}, \frac{29N}{129}), \tau_2^{(4)} = (\frac{13N}{96}, \frac{19N}{96}), \tau_2^{(5)} = (\frac{5N}{48}, \frac{11N}{48})$, while the second-order PHAF is calculated using $\tau_1^{(1)} = \frac{\pi}{4}, \tau_1^{(2)} = \frac{\pi}{2}, \tau_1^{(3)} = \frac{3\pi}{8}, \tau_1^{(4)} = \frac{5\pi}{8}, \tau_1^{(5)} = \frac{7\pi}{8}$.

Table 1. Phase coefficients of considered two-component PPS

<table>
<thead>
<tr>
<th>Component I</th>
<th>Component II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{1,1}$</td>
<td>$\frac{26}{27} N$</td>
</tr>
<tr>
<td>$a_{2,1}$</td>
<td>$\frac{(-53 + 22 \pi)}{125 N}$</td>
</tr>
<tr>
<td>$a_{2,2}$</td>
<td>$\frac{(-47 + 22 \pi)}{125 N}$</td>
</tr>
<tr>
<td>$a_{2,3}$</td>
<td>$\frac{22}{125 N}$</td>
</tr>
</tbody>
</table>

$\frac{9N}{16}, \tau_1^{(3)} = \frac{9N}{16}, \tau_1^{(4)} = \frac{11N}{40}, \tau_1^{(5)} = \frac{27N}{100} N$. All the time lags were rounded to the closest integer before use in the ml-HIM calculation. In the time lags selection, we followed the recommendation (10).

The simulation results are depicted in Fig. 1. In specific, we calculated the mean squared error (MSE) for each coefficient given in Table 1. The MSE is defined as

$$MSE = 10 \log_{10} \frac{\sum_{k,m=1}^{N_sim} (\hat{\alpha}_{k,m} - \alpha_{k,m})^2}{N_{sim}},$$

(12)

where $\hat{\alpha}_{k,m}$ is the estimation of the $n$th coefficient $(n = 1, 2, 3)$ of the $k$th component in the $n$th simulation, $\alpha_{k,m}$ is the true coefficient’s value, and $N_{sim}$ is the number of Monte Carlo simulations. In our simulations, $N_{sim} = 500$. The left and right sides of the figure correspond to the results obtained for the first and second components, respectively. The top, middle and bottom rows correspond to the estimation of the third-, second- and first-order coefficients, respectively. The proposed dichotomous search, carried out with $L = 7$, is compared to the estimations provided by interpolating the PHAF maximum bin, where we considered interpolations with $N_i = 1$ (no interpolation), $4$, $16$ and $64$. The results obtained for $N_i = 64$ practically coincide with the proposed method, however with $N_i/(L + 2) \approx 7.1$ times more calculations. The case $N_i = 1$ is significantly outperformed by all the other cases. In particular, compared to the case $N_i = 1$, the proposed method, with only 9 additional PHAF points, achieves the MSE improvement of around 20dB for all the estimated coefficients.

Further increasing of $L$ would not yield any significant improvement. The estimation bias due to the interaction among components limits the PHAF’s performance [4]. In the estimation of the second (weaker) component, however, the bias is smaller, which is due to the fact that prior to the estimation of weaker component, the stronger component was filtered out from the original signal. Fortunately, the bias can be removed by using an NLS approach. In [4], an NLS approach that uses the NMSA for minimizing the nonlinear cost function is proposed. The MSE that corresponds to the NMSA is also presented in Fig. 1 (depicted by $\times$). The price paid for improved accuracy, however, is significantly increased computational complexity. For example, the execution time of the NMSA exceeded the proposed method’s time around 60 times. Note that the NMSA is implemented in Matlab through the fminsearch function and it optimizes
the NLS cost function $f(\theta)$ that involves the matrix inversion (see [4, eq. (35)]). $f(\theta)$ mainly contributes to the algorithm’s complexity.

In [4], the algorithm is initialized with coefficient estimations provided by the PHAF. Herein, we applied the NMSA with initial estimates output by each of the methods considered in Fig. 1. We calculated the average number of iterations (over 100 runs) for each method and presented them in Fig. 2. Clearly, the more accurate initial estimates the faster the algorithm. In particular, our method provides a computational complexity decrease of around 20% compared to the PHAF interpolated with $N_i = 1$ and 4, and around 10% compared to $N_i = 16$. Therefore, having in mind the execution requirements of the NMSA, our method can provide substantial time savings.

5. CONCLUSIONS

We proposed a fine search algorithm for the PHAF-based mPPS parameter estimation. With calculation of several additional PHAF points we are able to estimate the peak frequency very accurately, thus preventing the need for oversampling.

Moreover, if the estimates output by the proposed method initialize an NLS-based estimation method, in particular the NMSA, the execution time can be significantly reduced.

6. REFERENCES


