ABSTRACT

The Restricted Isometry Property (RIP) is an important concept in compressed sensing. It is well known that many random matrices satisfy the RIP with high probability, whenever the entries of the random matrix have finite second order moment. Recent work in compressed sensing has shown that it is possible to do dimensionality reduction and signal reconstruction using Cauchy random projections. This suggests that the $l_1$ distance is preserved when one projects a set of data points from a high-dimensional space, to one of lower dimension with a random matrix which does not have finite variance. This paper generalizes this concept where it is shown that $\alpha$-stable projections, which preserve the $l_\alpha$ distance, also satisfy a generalized RIP property and consequently reconstruction from $\alpha$-stable projections is feasible.

Index Terms— Restricted Isometry Property, Fractional Lower Order Moments, $\alpha$-Stable Random Variables, Compressed Sensing

1. INTRODUCTION

Dimensionality reduction methods by linear random projections enable the mapping of a set of high-dimensional data points into a set of points in low-dimension, such that both sets have similar distance properties. The choice of the random projection matrix depends on which norm is preferred. Indyk [1] proposed constructing the random matrix from i.i.d. samples of $\alpha$-stable distributions, for dimension reduction in $l_\alpha$ ($0 < \alpha \leq 2$). This is something natural since a $\alpha$-stable projection matrix preserves the $l_\alpha$ distance\(^1\) of the set of high dimensional data points. In [2], Cauchy random projections are used to project a set of data points into a space of lower dimension, in which the distance between two points is estimated. Given that the Cauchy distribution is a 1-stable random variable, the projection matrix preserves the $l_1$ distance between two points in the high dimensional set. The importance of Cauchy random projections lies on the fact that the $l_1$ norm is more robust to noise, missing data, and outliers, than the $l_2$ norm. The $l_1$ distance estimation problem reduces to estimating the Cauchy dispersion parameter of the distribution of a small number of i.i.d. samples. The work in [2] elaborates on various nonlinear estimates of this dispersion parameter.

Signal reconstruction using Cauchy random-projections was achieved in [3], extending the theory of dimensionality reduction with the $l_1$ norm. In [3] not only the $l_1$ distances of the original data were preserved, but algorithms to reconstruct the original signals with negligible error were derived. Since Cauchy random projections have undefined second-order statistics, the large suite of reconstruction methods developed for compressive sensing fail. The methods developed in [3] were built on a rich class of robust regression algorithms recently developed for signal processing under stable models [4]. In particular, the reconstruction problem was formulated as an iterative coordinate-descent parameter estimation problem.

The works mentioned above, suggests that $\alpha$-stable random matrices also preserve the information of distance when a set of data points are projected into a space of low-dimension. However, due to the lack of a finite second order moment, the RIP for random matrices with finite variance does not hold. In [5], in order to determine how many Gaussian measurements are sufficient to recover a signal with $l_p$ minimization, an $l_p$ variant of the RIP is formulated. Although in [5] the authors just consider the Gaussian case, this variant of the RIP can be generalized for $\alpha$-stable distributions with $1 \leq \alpha \leq 2$.

The interest in $\alpha$-stable random projections arises since the $l_\alpha$ norm that they preserve may be more robust than the $l_2$ norm. Furthermore, previous work has shown that to recover sparse signals, $l_p$ minimization ($0 < p < 1$) requires fewer linear measurements than $l_1$ minimization. In this paper, we show a way to determine how many $\alpha$-stable measurements are needed for this $l_p$ variant of the RIP to hold with high probability.

2. $\alpha$-STABLE PROJECTIONS IN COMPRESSED SENSING

For symmetric $\alpha$-stable distributions ($S\alpha S$) with location parameter equal to zero, the characteristic function is given by

$$
\phi(t) = e^{-\gamma |t|^\alpha},
$$

where $\gamma$ is the dispersion of the $S\alpha S$ distribution. Since $e^{-\gamma |t|^\alpha}$ is an even function, the density function of an $S\alpha S$ is then defined by

$$
f(x) = \frac{1}{\pi} \int_0^\infty e^{-\gamma |t|^\alpha} \cos(tx)dt.
$$

It can be shown that $f(x)$ is given by

$$
f(x) = \begin{cases}
\frac{1}{\pi} \frac{1}{\gamma^{1+\alpha}} & \alpha = 1 \\
\frac{1}{\pi \gamma^{1+\alpha}} \sum_{n=0}^{\infty} (-1)^n \Gamma \left( \frac{2n+1}{\alpha} \right) \left( \frac{x}{\gamma} \right)^{2n} & 1 < \alpha < 2 \\
\frac{1}{\pi \gamma^{1+\alpha}} e^{\frac{x^2}{4\gamma}} & \alpha = 2.
\end{cases}
$$

An interesting property of $S\alpha S$ distributions is that the density function of a linear combination of i.i.d. $\alpha$-stable random variables is a

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\(^1\)The $l_\alpha$ distance that $\alpha$-stable random projections preserve is a norm if $1 \leq \alpha \leq 2$. 

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$S\alpha S$ random variable as well, but with a dispersion determined by the scalars of the linear combination and the dispersions of each one of the i.i.d. $S\alpha S$ random variables. This can be seen using the characteristic function. Define $Y$ as a linear combination of i.i.d. $S\alpha S$ random variables: $a_1X_1 + a_2X_2 + \cdots + a_nX_n = Y$. The characteristic function of $Y$ is equal to

$$\varphi_Y(t) = e^{-a_1^2\gamma_1^2|t|^p}e^{-a_2^2\gamma_2^2|t|^p}\cdots e^{-a_n^2\gamma_n^2|t|^p} = e^{-(a_1^2\gamma_1^2 + a_2^2\gamma_2^2 + \cdots + a_n^2\gamma_n^2)|t|^p}.$$  

This implies that $Y$ is a $S\alpha S$ random variable with the same distribution of the $X_i$, but with dispersion equal to $\gamma = (a_1^2\gamma_1^2 + a_2^2\gamma_2^2 + \cdots + a_n^2\gamma_n^2)^{\frac{1}{2}}$. A special case is when the $X_i$ have the same dispersion and this one is equal to one. In this case, the dispersion of $Y$ is the $\ell_\alpha$ norm of the scalars of the linear combination.

Nevertheless, all non-Gaussian stable distributions do not have finite variance, and for this reason, the common linear estimators cannot always give an idea of the dispersion of a $\alpha$-stable random variable. In [2], three non-linear estimators are proposed to estimate the dispersion of a Cauchy random variable: the median, the geometric mean and the maximum likelihood estimator. These ones have different performances, but all of them are useful to estimate the dispersion of, not only Cauchy, but all $S\alpha S$ random variables. However, one can use another estimator of the dispersion of a random variable: the fractional moment of order $p$. This one is defined as $E(|x|^p)$, and whenever $p < \alpha$, it can be used to estimate the dispersion of a $S\alpha S$ random variable. The fractional moment of order $p$ of a $S\alpha S$ random variable is given by

$$E(|x|^p) = \frac{\Gamma(1 - p/\alpha)}{\Gamma(1 - p)} \cos\left(\frac{\pi}{2}p\right)^\gamma.$$  

Using the following properties of the Gamma function

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin\pi z},$$

$$\Gamma(z + 1) = z\Gamma(z),$$

$$\sqrt{\pi}\Gamma(2z) = 2^{z-1/2}\Gamma(z)\Gamma(z + 1/2),$$  

one obtains that $E(|x|^p)$ is equal to

$$E(|x|^p) = \frac{2^{p+1}\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)} \Gamma\left(\frac{p + 1}{2}\right)^\gamma.$$  

This result can be found in [6]. It should be pointed out that (3) is valid for $0 < p < 2$ and $0 < \alpha < 2$, as long as $p < \alpha$.

From (3) the following non-linear estimator follows

$$E\left(\frac{1}{k} \sum_{i=1}^{k} |x_i|^p\right) = \frac{2^{p+1}\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)} \Gamma\left(\frac{p + 1}{2}\right) \gamma^p,$$

$$E\left(\|X\|_p^p\right) = \frac{2^{p+1}\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)} \Gamma\left(\frac{p + 1}{2}\right) \gamma^p,$$  

where $X$ is a vector of $k$ components, each one a realization of a $S\alpha S$ random variable.

The importance of the fractional moment of order $p$ is that can be directly related to the $l_p$ variant of the RIP proposed in [5], as it will be see shortly. Furthermore, a new RIP for $S\alpha S$ random matrices is important for reconstruction algorithms such as the ones that it can be found in [3], methods of robust statistics, non-convex compressed sensing, among others.

3. RIP for $\alpha$-Stable Random Matrices

To prove the main theorem of the paper we will use probabilistic methods. This is a natural approach since the $l_p$ variant of the RIP may be seen as the non-linear estimator showed in (4). As it will be see in the theorem, the $\alpha$-stable random projections preserve the $l_\alpha$ norm. This should not be surprising given what it was stated in Section 2. The theorem is as follows:

**Theorem 1.** Let $\Omega$ be an $M \times N$ matrix whose elements are i.i.d. alpha-stable realizations with $1 \leq \alpha \leq 2$, which is multiplied by a scalar which depends only on $M, \alpha$ and $p$. If $M \geq C_1k\log(N/k) + C_2k + C_3$ and $0 < p < \alpha/2$, then

$$(1 - \delta)\|x\|_\alpha^p < \|\Omega x\|_p^p < (1 + \delta)\|x\|_\alpha^p$$  

holds with probability exceeding $1 - 1/(N^{\eta})$.

The proof of the theorem is divided in three lemmas and a proof that gathers what it was obtained in the lemmas. The first lemma calculates the probability that the RIP fails for a given $x$ that belongs to $\mathbb{R}^k$ and $\|x\|_0 = k$. This implies working with a scaled submatrix of $\Omega$, which is associated with the given $x$. In the second lemma, the probability that the RIP fails for any $x \in \mathbb{R}^k$ is determined using the union bound. To obtain this result, the probability that the RIP fails for any $x$ that belongs to the unit ball induced by the $l_\alpha$ norm is calculated. Taking advantage of the fact that a unit ball is a compact space, which allows to express any $x$ that belongs to the unit ball as a series, the minimum value that $\delta$ must have is obtained. Then, taking any $x$ that is in the border of the unit ball, the result is generalized for any $x \in \mathbb{R}^k$. The probability that the RIP fails for any submatrix of a scaled version of $\Omega$ is obtained in the third lemma. Finally, the proof of the theorem is given.

**Lemma 1.** Let $T$ be a set of indices with $|T| = k (1 \leq k \leq N)$, denote by $X_T$ the set of discrete signals in $\mathbb{R}^k$ that are zero outside of $T$. Let $\Psi$ be an $M \times k$ submatrix of $\Phi$ formed with the columns whose indices are in $T$. If $0 < p < \alpha/2$, $\eta > 0$ and for an $x \in X_T$ then

$$(1 - \eta)MC_\alpha(\alpha, p)\|x\|_\alpha^p < \|\Psi x\|_p^p < (1 + \eta)MC_\alpha(\alpha, p)\|x\|_\alpha^p$$  

with probability lower-bounded by $1 - 2e^{-\frac{1}{2}M\eta^2C_\alpha(\alpha, p)}$, where $C_\alpha(\alpha, p)$ is a constant which depends only on $\alpha$ and $p$.

**Proof:** Let $y = \Psi x$, where $x$ is a vector that belongs to $X_T$. The resultant vector $y$ has $M$ components, each one given by

$$y_i = \sum_{j \in T} \psi_{ij}x_j \sim S\alpha S(0, \|x\|_\alpha).$$

The $l_p^p$ norm of $y$ is equal to

$$\|y\|_p^p = \|\Psi x\|_p^p = \sum_{i=1}^{M} \sum_{j=1}^{k} |\psi_{ij}x_j|^p = \sum_{i=1}^{M} |y_i|^p.$$  

This implies that in average $\|\Psi x\|_p^p$ tends to

$$E(\|\Psi x\|_p^p) = M\frac{2^{p+1}\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)} \Gamma\left(\frac{p + 1}{2}\right) \|x\|_\alpha^p$$  

for $0 < p < \alpha$. If $0 < p < \alpha/2$, the variance of the estimator
\[ \|\Psi x\|_p^p \text{ is finite as well} \]

\[
Var(\|\Psi x\|_p^p) = E(\|\Psi x\|_p^2)^p - E(\|\Psi x\|_p^p)^2 \\
= MC_m(\alpha, 2p)\|x\|_{\alpha}^{2p} - M^2C_m(\alpha, p)^2\|x\|_{\alpha}^{2p}
\]

\[
Var(\|\Psi x\|_p^p) = (MC_m(\alpha, 2p) - M^2C_m(\alpha, p)^2)\|x\|_{\alpha}^{2p}.
\]

To obtain the density function of \(\|\Psi x\|_p^p\), one can use the central limit theorem to approximate this distribution with \(M\) sufficiently large. This is possible because, if \(p < 1/2\), each \(\|y_t\|_p^p\) has finite variance and mean. The distribution may be approximated with a Gaussian density function with the following parameters

\[
\mu = MC_m(\alpha, p)\|x\|_{\alpha}^p
\]

\[
\sigma^2 = M(C_m(\alpha, 2p) - C_m(\alpha, p)^2)\|x\|_{\alpha}^{2p}
\]

Using the Chernoff bound, it can be proved that for a Gaussian distribution the following inequality holds

\[
P(\|X - \mu\| < \eta \mu) \geq 1 - 2e^{-\frac{\eta^2 \mu^2}{2\sigma^2}}. \tag{6}
\]

Considering \(\|\Psi x\|_p^p\) as the random variable \(X\), and replacing by \(\mu\) and \(\sigma^2\) in (6) leads to

\[
P(\|\Psi x\|_p^p - E(\|\Psi x\|_p^p) < \eta E(\|\Psi x\|_p^p)) \geq 1 - 2e^{-\frac{\eta^2 M C_m(\alpha, p)^2}{2\sigma^2}} \geq 1 - 2e^{-\frac{\eta^2 M^2 C_m(\alpha, p)^2}{2\sigma^2}}
\]

where \(C(\alpha, p)\) is equal to \(C_m(\alpha, p)^2/C_m(\alpha, p)\). It is clear that the probability that (5) fails for a \(x \in X_T\) is \(P_{\alpha, p, M}(\eta) = 2e^{-\frac{\eta^2 M^2 C_m(\alpha, p)^2}{2\sigma^2}}\),

**Lemma 2.** Let \(0 < p < \alpha/2\), \(\Psi\) an \(M \times k\) submatrix of \(\Phi\) as in Lemma 2. Let \(\delta > 0\) and choose \(\epsilon, \eta > 0\) such that \(\frac{\epsilon^2 p + \eta}{1/p} = \delta\). Then for any \(x \in \mathbb{R}^k\)

\[
(1 - \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p < \|\Psi x\|_p^p < (1 + \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p \tag{7}
\]

holds with probability exceeding \(1 - (1 + 2/\epsilon)^k P_{\alpha, p, M}(\eta)\).

**Proof:** Let \(S\) be the unit ball of the \(l_\alpha\) norm in \(\mathbb{R}^k\). Let \(A\) be an \(\epsilon - \)net of \(S\) (with respect to the \(l_\alpha\) norm) having at most \(1 + 2/\epsilon\) \(k\) points\(^2\). Then, using the union bound, the probability that (7) fails for any \(x \in A\) is at most \((1 + 2/\epsilon)^k P_{\alpha, p, M}(\eta)\).

Let \(x \in S\). Then one can find \(x_0 \in A\) such that \(\|x - x_0\|_\alpha \leq \epsilon\). Letting \(\epsilon_1 = \|x - x_0\|_\alpha\), one has that \(\|x - x_0\|_\alpha \leq \epsilon_1\). Then one can find \(x_1\) such that \(\|x - x_0 - x_1\|_\alpha \leq \epsilon_1\). Continuing in this fashion, one can obtain sequences \((\epsilon_n)\) and \((x_n)\) such that \(\|x_n\|_\alpha \leq \epsilon^n\), and \(\|x - \sum_{n=0}^N e_n x_n\|_\alpha \leq \epsilon^{N+1}\), where \(\epsilon_0 = 1\) for convenience; therefore \(x = \sum_{n=0}^\infty e_n x_n\). Knowing that (7) holds for each \(x_n\) and that \(x_n \in S\), then

\[
\|\Psi x\|_p^p \leq \sum_{n=0}^\infty \epsilon_n \|\Psi x_n\|_p^p \leq \sum_{n=0}^\infty \|\epsilon_n \epsilon \|\Psi x_n\|_p^p \leq \sum_{n=0}^\infty \|\epsilon \|\Psi x_n\|_p^p \\
\leq \left(1 + \frac{\eta(1 + \epsilon)}{1 - \epsilon}\right) M C_m(\alpha, p)\|x\|_{\alpha}^p
\]

Also,

\[
\|\Psi x\|_p^p \geq \left(1 - \frac{\eta + \epsilon}{1 - \epsilon}\right) M C_m(\alpha, p)\|x\|_{\alpha}^p.
\]

For an arbitrary \(x \neq 0\), \(\|x/\|x\|_\alpha \in S\), then

\[
(1 - \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p < \|\Psi x/\|x\|_\alpha\|_p^p < (1 + \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p
\]

\[
(1 - \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p < \|\Psi x/\|x\|_\alpha\|_p^p < (1 + \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p
\] \tag{8}

**Lemma 3.** Let \(\Phi\) be an \(M \times N\) matrix, \(x \in \mathbb{R}^N\) and \(\|x\|_\alpha = k\).

Let \(0 < p < \alpha/2\) and \(\delta > 0\), then

\[
(1 - \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p < \|\Phi x\|_p^p < (1 + \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p
\]

\[
(1 - \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p < \|\Phi x\|_p^p < (1 + \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p
\]

with probability lower bounded by \(1 - (eN/k)^k(1 + 2/\epsilon)^k P_{\alpha, p, M}(\eta)\).

**Proof:** The probability that any \(M \times k\) submatrix of \(\Phi\) fails to satisfy (8) is \((\epsilon)^k(1 + 2/\epsilon)^k P_{\alpha, p, M}(\eta)\). However, \((\epsilon)^k \leq (eN/k)^k\).

This leads to

\[
\left(\frac{eN}{k}\right)^k(k + 2/\epsilon)^k P_{\alpha, p, M}(\eta).
\]

**Proof of Theorem 1.** According to Lemma 3 one has

\[
(1 - \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p < \|\Phi x\|_p^p < (1 + \delta)M C_m(\alpha, p)\|x\|_{\alpha}^p
\]

\[
(1 - \delta)\|x\|_{\alpha}^p < \|\Phi x\|_p^p < (1 + \delta)\|x\|_{\alpha}^p
\]

Letting \(\Omega\) equal to \(\frac{1}{M C_m(\alpha, p)^{1/p}}\Phi\) one obtains

\[
(1 - \delta)\|x\|_{\alpha}^p < \|\Omega x\|_p^p < (1 + \delta)\|x\|_{\alpha}^p.
\]

The probability that \(|\Omega|\) fails to satisfy (8) is \((\epsilon^2)^k(1 + 2/\epsilon)^k P_{\alpha, p, M}(\eta)\). One wants this quantity to be bounded above by \(1/(\epsilon)^k \geq (k/eN)^k\),

\[
\left(\frac{k}{eN}\right)^k \geq 2 \left(\frac{eN}{k}\right)^k(k + 2/\epsilon)^k P_{\alpha, p, M}(\eta).
\]

\[
\frac{1}{2} M^2 C_m(\alpha, p) \geq e^{2k \log(eN/k) + k \log(1 + 2/\epsilon) + \log 2}.
\]
Therefore,
\[
\frac{1}{2} M \eta^2 C(\alpha, p) \geq 2k \log(eN/k) + k \log(1 + 2/\epsilon) + \log 2.
\]
\[
M \geq \frac{4}{\eta^2 C(\alpha, p)} k \log(N/k) + \log 4.
\]
\[
\frac{4}{\eta^2 C(\alpha, p)} (\log \sqrt{1 + 2/\epsilon} + 1) k.
\]
\[
M \geq C_1 k \log(N/k) + C_2 k + C_3 \epsilon
\]

It should be noted that as \( p \to 0 \), \( C(\alpha, p) \to \infty \), then \( C_1 \), \( C_2 \) and \( C_3 \) tend to zero. Therefore, \( M \) should be greater or equal than zero as \( p \) tends to zero. However, since the density function of \( \|\Psi x\|_p \) was approximated with a Gaussian distribution, this result is not very accurate for small \( M \). Nevertheless, as \( p \to 0 \), less measurements are required for the RIP to hold with high probability.

### 4. CONCLUSIONS

A new RIP for SaS random matrices with \( 1 \leq \alpha \leq 2 \) has been formulated. This is an important result for applications such as methods of robust statistics, non-convex compressed sensing, reconstruction algorithms [3], among others. Also, the main theorem of the paper shows that the number of measurements decreases as \( p \to 0 \). However, despite that equation (4) is valid for \( 0 < \alpha < 2 \), the RIP can not be generalized for \( 0 < \alpha < 1 \). The reason for this is that the Theorem 2 states that the unit ball must be induced by a norm. Since the \( l_{\alpha} \) distance is a norm for \( \alpha = 1 \), the theorem is not valid for \( 0 < \alpha < 1 \). It is possible that a formulation of \( \epsilon - \) nets for non-convex bodies, such as the unit balls induced by the \( l_{\alpha} \) distances with \( 0 < \alpha < 1 \), leads to a generalization of this RIP for these values of \( \alpha \).

### 5. APPENDIX A

A \( \epsilon \)-net is a finite covering of a unit ball induced by certain norm, with unit balls that are induced by another norm which have a radius less than \( \epsilon \). The center of the unit balls that compose the covering are located in such a way that the distance between one of these centers, and any other point that belongs to the unit ball that is being covered, is less than \( \epsilon \). In [7] the following result can be found:

Let \( \| \|_1 \) and \( \| \|_2 \) be two arbitrary norms on \( \mathbb{R}^n \) with respective unit balls \( B_1 \) and \( B_2 \). Let \( \epsilon > 0 \), then there is a finite set \( A \subseteq B_1 \) with

\[
\text{card}(A) \leq \frac{\text{vol}(2/\epsilon) B_1 + B_2}{\text{vol}(B_2)}
\]

which is a \( \epsilon \)-net for \( B_1 \) with respect to \( \| \|_2 \), that is to say \( \forall x \in B_1 \exists y \in A \) such that \( \|x - y\|_2 < \epsilon \).

This result is important to prove the following theorem:

**Theorem 2.** Let \( B \) be the unit ball of the \( l_{\alpha} \) norm in \( \mathbb{R}^n \). Let \( A \) be an \( \epsilon \)-net of \( B \) (with respect to the \( l_{\alpha} \) norm). \( \forall x \in B \exists y \in A \) such that \( \|x - y\|_\alpha < \epsilon \) and \( A \) has at most \( (1 + 2/\epsilon)^n \) points.

**Proof:** Let \( \| \|_1 = \| \|_2 = \| \|_\alpha \). Let \( B \) be the unit ball of the \( \| \|_\alpha \) norm. Then (10) becomes

\[
\text{card}(A) \leq \frac{\text{vol}(2/\epsilon) B + B}{\text{vol}(B)}
\]

\[
\leq \frac{\text{vol}(2/\epsilon + 1) B}{\text{vol}(B)}.
\]

\[
\leq \frac{\text{vol}(B)(2/\epsilon + 1)^n}{\text{vol}(B)}.
\]

\[
\text{card}(A) \leq (2/\epsilon + 1)^n.
\]

The Theorem 2 follows.

### 6. APPENDIX B

It should be noted that

\[
C(\alpha, p) = C_\mu(\alpha, p)^2 C_{\sigma^2}(\alpha, p) = \frac{1}{C_\mu(\alpha, 2p) C_{\sigma^2}(\alpha, p)} - 1.
\]

Using (2) it can be shown that

\[
C_\mu(\alpha, 2p) C_{\sigma^2}(\alpha, p) = \frac{\alpha \sqrt{\pi}}{2^{p/\alpha + p/2 + 1}} \Gamma(-p/2) \Gamma(-p/\alpha + 1/2) \Gamma(\frac{p + 1}{2})
\]

To know to what value tends \( C(\alpha, p) \) as \( p \to 0 \), it is necessary to know the behavior of the term \( \Gamma(-p/2) \Gamma(-p/\alpha) \) as \( p \) tends to zero. This can be done using the following property of the Gamma function

\[
\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z + 1)(z + 2) \ldots (z + n)} \tag{11}
\]

Using (11) it is easy to see that

\[
\Gamma(-p/2) \Gamma(-p/\alpha) = \lim_{n \to \infty} \frac{(-p/\alpha)n^{-p/2}(-p/\alpha + 1) \ldots (-p/\alpha + n)}{(-p/2)n^{-p/2}(-p/2 + 1) \ldots (-p/2 + 2n)}
\]

This implies that

\[
\lim_{p \to 0} \frac{\Gamma(-p/2)}{\Gamma(-p/\alpha)} = \frac{2}{\alpha}
\]

Therefore, \( \lim_{p \to 0} C(\alpha, p) = \infty \).

### 7. REFERENCES


