Semi-definite Programming for Distributed Tracking of Dynamic Objects by Nonlinear Sensor Network

U. Rashid\(^1\), H.D. Tuan\(^1\), H.H. Kha\(^1\) and H.H. Nguyen\(^2\)

\(^1\)University of New South Wales, Sydney, Australia. Email: \{umar,h.d.tuan,h.k.ha\}@unsw.edu.au
\(^2\)University of Saskatchewan, Saskatoon, Canada. Email: ha.nguyen@usask.ca.

Abstract—This paper discusses dynamic state estimation for nonlinear measurement model through distributed multisensor network under power constraints. For this scenario, we propose an optimized power allocation strategy based on semidefinite programming, that achieves minimum mean-squared error for the estimate subject to constraints on total transmit power. System nonlinearity is handled effectively with the help of distributed unscented Kalman filtering and linear fractional transformation. Furthermore, advantage of using multiple sensors over a single independent sensor is established through simulation results for tracking a maneuvering target.

Index terms. Nonlinear sensor network, unscented transformation, semi-definite programming, distributed linear fractional transformation filtering

I. INTRODUCTION

Wireless sensor network (WSN) is an ever emerging technology which has the potential of being used in many applications like environment sensing, traffic monitoring, military surveillance [1], [2], [6]. Nonlinear sensor networks (NSNs) practically arise when sensors’ observations bear a nonlinear relationship with the parameter to be estimated. Typically such a network consists of multitude of tiny inexpensive sensor nodes deployed randomly or deterministically over an area of interest. Each node makes its own local observation independently, while a central unit, called fusion center (FC), monitors the entire network to ensure high-quality performance. Fusion of noisy observations from these spatially distributed nodes delivers a more reliable picture of the environment that is difficult to achieve with a single independent sensor.

The sensor nodes are characterized by their low power profiles and limited computational capability. Their power requirements become pronounced when they have to transmit their observations towards FC through a noisy channel. Therefore, an algorithm must be found to achieve energy-efficient operation of sensor nodes.

Based on the knowledge of unconditional mean of a random state variable, conditioning random observations, covariance of the conditioning random variable and cross-covariance of measurement and state variables, the FC can compute linear minimum mean square error (LMMSE) estimate using recursive Bayes’ filtering. This works well for systems bearing linear relationships in state space as well as measurement model description. For nonlinear systems, we have to resort to some linearizing approximations to estimate the evolving state, since most of the physical systems involve nonlinear mappings.

Earlier reported work concerning estimation in nonlinear dynamic models includes various Kalman-filter-based methods, such as the Extended Kalman Filter (EKF), interacting multiple model EKF, all of which are based on the assumption that the state estimate lies in neighborhood of the mean value. Unscented Kalman Filter (UKF) [4], which relies on statistical linear regression technique, has been shown to offer better approximations [5], [12] under nonlinear models than EKF. However, a recently proposed technique based on using linear fractional transformation (LFT) along with UKF, offers even better estimation results for nonlinear models such as target tracking problems [8]. In this paper, we extend the target tracking problem to a multisensor case in an energy efficient manner.

Notion of distributed Kalman filtering, in conjunction with consensus filters, for target tracking has been addressed in [7], [11]. These papers employ scalable Kalman filters across highly computational sensor nodes capable to communicate with their neighboring nodes. However, in an energy-constrained environment, this idea does not seem to provide efficient tracking performance. On the other hand, our proposed filtering algorithm, when combined with multisensor framework, addresses target tracking problem in a highly energy efficient manner.

The objective of this paper is to seek an efficient power allocation scheme for sensor nodes which aims at achieving minimum mean square error (MSE) of the state estimate subject to total transmit power constraints for a dynamic nonlinear measurement model. This problem is subsequently cast as a standard semidefinite program (SDP) of tractable optimization which ensures global optimal solution. In our previous paper, we presented MMSE estimation for static measurement model in nonlinear sensor network [10]. In short, the key contributions of the paper are two-fold:

- It applies optimized power allocation strategy for dynamic state estimation under a nonlinear measurement model.
- It further shows that estimation/tracking performance of multisensor network is better than a single independent sensor even in an energy-constrained scenario.

The rest of the paper is organized as follows. Section II describes system model and formulation of an optimization problem as a semidefinite program (SDP). Section III presents procedure for handling by applying LFT. Section IV discusses
Most of the notations used in the paper are fairly standard. Set of real numbers is denoted by $\mathbb{R}$. Notation used for a positive semi-definite matrix is $A \succeq 0$. A square $n \times n$ diagonal matrix is represented by $\text{diag}(a_1, a_2, \ldots, a_n)$. Expectation of a random variable $x$ is represented as $\mathbb{E}[x]$. Trace of a square matrix $A$ is expressed as $\text{tr}(A)$. Autocorrelation of vector $x$, and cross-correlation between $x$ and $y$ are denoted by $R_x = \mathbb{E}\{xx^T\}$ and $R_{xy} = \mathbb{E}\{xy^T\}$, respectively.

II. PROBLEM FORMULATION AND SDP BASED SOLUTION

In what follows, for two random variables (RVs) $x, y$ with the expectation/means $\mathbb{E}\{x\} = \bar{x}, \mathbb{E}\{y\} = \bar{y}$, its covariance is $\text{Cov}(x, y) = \mathbb{E}\{(x - \bar{x})(y - \bar{y})^T\}$. The most important question is how to estimate $x$ by an affine function $A_ky + b_k$:

**Theorem 1:** It is true that

$$
R_{yx}^T, \bar{x} - R_{yx}^T \bar{y} = \arg\min_{A_k, b_k} \mathbb{E}\{|x - (A_ky + b_k)|^2\}
$$

(1)

Consequently, the LMMSE estimation of $x$ based on the observation $\bar{y} = y$ for any random variables $x$ and $Y = \bar{x} + R_{yx}^T \bar{y} + \epsilon$, where the RV error $\epsilon = x - \bar{x} - R_{yx}^T \bar{y}$ is always uncorrelated with $y$ ($R_{ey} = R_{yx}^T - R_{yx}^T R_{yy}^{-1} R_{ey} = 0$) and is zero means with the auto-covariance $P_{\epsilon} = R_{e} - R_{ey} R_{yy}^{-1} R_{ey}$.

Consider state space representation of a dynamic sensor model in a network consisting of $N$ nodes at the $k$th time instant

$$
\theta_{k+1} = g(\theta_k) + v_k, \quad y_k = f(\theta_k) + n_k, \quad z_k = A_k y_k + w_k,
$$

(2)

where $g$ and $f$ denote nonlinear mappings, $v_k \sim \mathcal{N}(0, R_v)$ and $n_k \sim \mathcal{N}(0, R_n)$ are mutually independent process noise and measurement noise which are also independent of $\theta_k$. Here $\theta_k \in \mathbb{R}^M, y_k \in \mathbb{R}^N$ are system state variable and sensor measurement, respectively. $z_k$ is the measurement vector received at FC through a relay matrix $A_k \in \mathbb{R}^{N \times N}$ defined by $A_k = \text{diag}(\sqrt{\alpha_1 \sqrt{h_1}}, \ldots, \sqrt{\alpha_N \sqrt{h_N}})$ with channel gain $\sqrt{h_i}$ and amplification coefficient $\sqrt{\alpha_i}$.

Suppose $\hat{\theta}_{k|k-1}$ is the estimator of $\theta_k$ by using $k-1$ past measurements $z_0, z_1, \ldots, z_{k-1}$. At time instant $k$, the filtering problem at FC involves two steps:

- Upon receiving observations from all sensor nodes, FC performs the optimal estimation $\hat{\theta}_{k|k} = \theta_k|z_k$ by using optimal amplification coefficients $\alpha_k$ in the equation

$$
y_k = f(\hat{\theta}_{k|k-1}) + n_k, \quad z_k = A_k y_k + w_k
$$

(3)

Using Theorem 1 and the inverse matrix lemma, estimation error covariance matrix $R_{ek} := R_{\theta_{k|k} - \theta_{k|k-1}}$ is given by

$$
R_{\theta_{k|k-1}} - R_{\theta_{k|k-1} y_k} R_{y_k} R_{\theta_{k|k-1} y_k}^T + R_{\theta_{k|k-1} y_k} (R_y + R_{y_k} A_k^T R_{\theta_{k|k}} A_k) R_{\theta_{k|k-1} y_k}^{-1} R_{\theta_{k|k-1} y_k}^{-1}
$$

where $A_k = \text{diag}(a_1, a_2, \ldots, a_N)$ and $R_y = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$. Thus, the FC’s objective is to find the diagonal matrix $A_k$ for minimizing the MMSE $\mathbb{E}\{|\hat{\theta}_{k|k} - \theta_{k|k-1}|^2\} = \text{tr}(R_{ek})$ under the budget constraint $\text{tr}(A_k R_y A_k^T) \leq P_{\text{total}}$, which can be expressed by the following SDP via the variable change $x_{\alpha} = A_k^T A_k [10]$

$$
\min_{G, x_{\alpha}} \text{tr}(G) \text{ s.t } x_{\alpha} \succeq 0, \text{ tr}(X_{\alpha} R_y) \leq P_{\text{total}}.
$$

- Apply Theorem 1 to the state equation

$$
\theta_{k+1|k} = g(\theta_{k|k}) + v_k
$$

(4)

to update $\theta_{k+1|k}$. Like the above SDP, this require calculation of $R_{\theta_{k+1} y_k}$ as well as $\theta_{k+1|k}$.

III. LINEAR FRACTIONAL TRANSFORMATIONS FOR COVARIANCE APPROXIMATION

One can note that in order to implement the above SDP for optimally updating $\theta_{k|k}$ from (3) and to update $\theta_{k+1|k}$ from (4), we need to calculate $R_v$ and $R_{\theta y}$ from a nonlinear equation

$$
y = F(\theta) + n
$$

(5)

where either $F = f, \theta = \theta_{k|k-1}, n = n_k$ (for (3)) or $F = g, \theta = \theta_{k|k}, n = v_k$ (for (4)).

When $F$ is a quadratic map, such calculations can be effectively implemented through the so called unscented transformation [4]. However, for $F$ of either higher order forms or involving fractional terms, the Linear Fractional Transformation (LFT) is more useful [8] as described below.

Firstly, any smooth nonlinear map $F$ can be equivalently represented by an LFT model [8]

$$
\begin{bmatrix}
y \\
y_{\Delta}
\end{bmatrix} =
\begin{bmatrix}
C_1 & D_1 \\
C_2 & D_2
\end{bmatrix}
\begin{bmatrix}
\theta \\
\omega_{\Delta}
\end{bmatrix} +
\begin{bmatrix}
n \\
0
\end{bmatrix}, \quad \omega_{\Delta} = \Delta(\theta)y_{\Delta},
$$

(6)

where $C_1, D_i$ are deterministic matrices and the simple nonlinear feedback $\omega_{\Delta} = \Delta(\theta)y_{\Delta}$ captures all nonlinearity of $F$. Thus, the approximation is localized to this feedback path for estimation of auxiliary variable $\omega_{\Delta}$. Expectation of $y$ is approximated by

$\bar{y} = C_1 \bar{\theta} + D_1 \bar{\omega}_{\Delta}$

where

$$
\bar{\omega}_{\Delta} = (I - \Delta D_2)^{-1} \left( \frac{1}{p!} \sum_{j=0}^{p} \Delta(\theta(j)) C_2 y(j) \right),
$$

$\Delta = \frac{1}{p+1} \sum_{j=0}^{p} \Delta(\theta(j))$. Here $\theta(j)$ are sigma points evaluated for $M$-dimensional $\theta$ by statistical linear regression as

$$
\theta(j) = \bar{\theta}(0) + \sqrt{\frac{p+1}{2}} R_j, \quad \theta(j+M) = \bar{\theta}(0) - \sqrt{\frac{p+1}{2}} R_j.
$$

Since $R_\theta \succ 0$, it follows Cholesky decomposition, $R_\theta = \sum_{j=0}^{M} R_j R_j^T, \bar{\theta}(0) = E[\theta_0]$ and $p = 2M$. Let $y(j) = F(\theta(j)), j = 0, 1, \ldots, p$, then according to (6), $\omega_{\Delta} =$
Δ(θ(𝑗))𝑦(𝑗). Covariance of 𝑦 and its cross-covariance with θ are approximated by

\[ R_y = C_1 R_0 C_1^T + D_1 R_Δ D_1^T + R_n \]
\[ R_{yθ} = C_1 R_0 + D_1 R_Δ \]

respectively, where

\[ R_Δ = \frac{1}{p+1} \sum_{j=0}^{p} (\omega_Δ(𝑗) - \bar{\omega}_Δ)(\omega_Δ(𝑗) - \bar{\omega}_Δ)^T, \]
\[ R_Δθ = \frac{1}{p+1} \sum_{j=0}^{p} (\omega_Δ(𝑗) - \bar{\omega}_Δ)(θ(𝑗) - \bar{θ})^T. \]

IV. SIMULATION RESULTS

The estimation of state variable in nonlinear measurement system using UKF and LFT is illustrated by considering problem of target tracking based on direct measurement of range and bearing. Such a model admits following set of state and measurement equations:

\[ \theta_{k+1} = A\theta_k + v_k \]
\[ y_{ik} = \left( \sqrt{(s_{i,x} - p_{x_k})^2 + (s_{i,y} - p_{y_k})^2} \right) \arctan \left( \frac{s_{i,y} - p_{y_k}}{s_{i,x} - p_{x_k}} \right) + n_i, \]

where \((s_{i,x}, s_{i,y})\) denote the \(i\)th sensor coordinates and \((p_{x_k}, p_{y_k})\) are coordinates of the target at the \(k\)th time instant. Here

\[ A = \begin{pmatrix} 1 & \sin \omega T & 0 & -1 - \cos \omega T \\ 0 & \cos \omega T & 0 & -\sin \omega T \\ 0 & -\cos \omega T & 1 & \sin \omega T \\ 0 & \sin \omega T & 0 & \cos \omega T \end{pmatrix}, \quad \theta_k = \begin{pmatrix} p_{x_k} \\ p_{y_k} \\ p_{x_k} \\ p_{y_k} \end{pmatrix} \]

account for a coordinated turn model [3]. Turn rate is denoted by \(\omega\) and sampling period by \(T\). For this model, conventional unscented transformations are applied to handle range measurement [9] whereas LFT method is employed to handle nonlinearity of the bearing information. Equivalent LFT representation for the above bearing-measurement model is specifically given in [8] for a single sensor case as

\[ C_{1i_k} = 0_{1 \times 4}, \quad D_{1i_k} = \frac{1}{s_{i,x}} [s_{i,y} - s_{i,x}] \]
\[ C_{2i_k} = 0_{1 \times 4}, \quad D_{2i_k} = \frac{1}{s_{i,x}} [1 - s_{i,y}] \].

The problem deals with tracking a maneuvering target moving within 2-dimensional surveillance region of \([0, 0] \times [500, 200]\)m². A multitude of sensor nodes \(N = 10\), randomly deployed over the region of interest, takes measurements at a sampling period of 1sec. For process noise, we choose \(\sigma_\omega^2 = 0.5\). Similarly, variance of local as well as global channel noise while measuring range and bearing are \(\sigma_n^2, range = 0.5, \sigma_n^2, bearing = \frac{\sigma_n^2}{100}\) and \(\sigma_n^2, range = 1.0, \sigma_n^2, bearing = \frac{\sigma_n^2}{100}\), respectively. However, this noise is scaled differently for every sensor so that every sensor has nonidentical channel conditions. Target’s trajectory, along with sensor nodes dis-

![Fig. 1. Mean Square Error in estimate of a maneuvering target coordinates from 1-10 sec.](image-url)
V. CONCLUSION

An optimized power allocation scheme is proposed for tracking a dynamic target in energy-constrained environment. Nonlinearity of the system is handled by combination of distributed unscented Kalman filtering and linear fractional transformation. An optimization problem formulated by minimizing mean-squared error of the estimate of target’s trajectory subject to constraint on total transmit power is then cast as a semidefinite program. Simulation results are provided to demonstrate that even with limited energy resources multisensor network offers better tracking performance than a single sensor.

REFERENCES