AN APPROACH OF DOA ESTIMATION USING NOISE SUBSPACE WEIGHTED $\ell_1$ MINIMIZATION

Chundi Zheng$^{1,2}$, Gang Li$^1$, Hao Zhang$^1$, Xiqin Wang$^1$

1. Department of Electronic Engineering, Tsinghua University, Beijing 100084, China
2. Naval Arms Command College, Guangzhou 510430, China

ABSTRACT

Using multiple measurement vectors (MMV), we propose an algorithm based on weighted $\ell_1$ minimization for direction-of-arrival (DOA) estimation, in which the weights are obtained by exploiting the orthogonality between the noise subspace and the array manifold matrix. The proposed algorithm penalizes the nonzero entries whose indices correspond to the row support of the jointly sparse signals by smaller weights and the other entries whose indices are more likely to be outside of the row support of the jointly sparse signals by larger weights, and therefore it can encourage sparsity at the true source locations. Numerical examples prove that the proposed algorithm has better performance than existing algorithms based on regular $\ell_1$ minimization.

Index Terms—Direction-of-arrival estimation, sparse signal recovery, weighted $\ell_1$ minimization, array processing

1. INTRODUCTION

In practice the sparse signal recovery is usually carried out by $\ell_1$ norm minimization due to it is a convex problem. However, $\ell_1$ norm minimization has a disadvantage that larger coefficients of signal are penalized more heavily than smaller coefficients, unlike the more democratic penalization of the $\ell_0$ norm [1]. This incurs the degradation of signal recovery performance based on regular $\ell_1$ minimization [1]. To conquer this problem, the iterative reweighted $\ell_1$ minimization was designed for the single measurement vector (SMV) problem, in which large weights could be used to discourage nonzero entries in the recovered signal [1]. The convergence of the iterative reweighted $\ell_1$ minimization was elaborated in [2]. Xu et al provided the theoretical results that the iterative reweighted $\ell_1$ minimization can indeed improve recoverable sparsity thresholds upon the regular $\ell_1$ minimization in the noiseless case [3]. Needell proved that the iterative reweighted $\ell_1$ minimization can improve the recovery accuracy in the noisy case [4]. The essence of the iterative reweighted $\ell_1$ minimization algorithm is that large weights could be used to banish the entries whose indices are more likely to be outside of the signal support, which promotes sparsity at the right positions [1-4].

In this paper, we extend the methodology of the iterative reweighted $\ell_1$ minimization from the SMV case [1] to the multiple measurement vectors (MMV) case for direction-of-arrival (DOA) estimation. We exploit the orthogonality between the noise subspace and the array manifold matrix [5-6] to achieve the idea of weighted $\ell_1$ minimization: the nonzero entries whose indices are inside of the row support of the jointly sparse signals are penalized by smaller weights and the other entries whose indices are more likely to be outside of the row support are penalized by larger weights. We call the proposed algorithm as Noise Subspace Weighted $\ell_1 - \ell_2$ algorithm (NSW- $\ell_1 - \ell_2$). Compared with the iterative reweighted $\ell_1$ minimization algorithm that focuses on the SMV case [1], forming the noise subspace weights neither uses iterative process nor needs an application-dependent parameter. The experiments prove that the NSW- $\ell_1 - \ell_2$ algorithm can achieve better performance than $\ell_1$-SVD algorithm [7-8] which exploits straightforwardly $\ell_1$ minimization.

The remainder of this paper is organized as follows. In the next section, we describe the problem of DOA estimation in the sparse signal framework. In Section 3, we introduce how to implement the NSW- $\ell_1 - \ell_2$. In Section 4, the performance of proposed method is explored by some examples. The summarization is given in Section 5.

2. BACKGROUND

We consider an array composed of $M$ sensors and suppose that there are $p$ far-field narrowband signals impinging on
the array from distinct directions \( \{ \theta_k, \, k = 1, 2, \cdots, p \} \). The measurements corrupted by additive noise can be described with the MMV form

\[
Y = AS + N, \quad (1)
\]

where \( Y = [y(1), \cdots, y(T)] \), \( S = [s(1), \cdots, s(T)] \) and \( N = [n(1), \cdots, n(T)] \) are the matrix of the received signals, the incident signals and the additive noise, respectively. \( T \) is the total number of snapshots. The matrix \( A = [a(\theta_1), \cdots, a(\theta_p)] \in \mathbb{C}^{M \times p} \) is the array manifold matrix, where the vector \( a(\theta_k) \) is called steering vector. The vector \( n \) is an additive Gaussian white noise vector with zero-mean and the variance \( \sigma^2 \). Without loss of generality, the signal and the noise are assumed to be uncorrelated.

It is a convincing viewpoint that the measurements \( Y \) can be seen as the sum of the signals from all possible spatial directions, i.e., \( Y = \Phi X + N \), where \( X \) denotes the jointly sparse signals, \( \Phi = [a(\phi_1), \cdots, a(\phi_K)] \in \mathbb{C}^{M \times K} \) is an overcomplete basis matrix [7-9], i.e., \( M < K \), the angles set \( \{ \phi_1, \cdots, \phi_K \} \) denotes a sampling grid of all possible source locations. The row support of the jointly sparse signals \( X \) can be defined as \( \text{Supp}_r(X) = \{ i : X_{i}^{(j)} \neq 0 \} \subseteq \Lambda \) [10], where \( X_{i}^{(j)} \) denotes the \( j \)-th entry of \( X^{(i)} \), \( X^{(i)} \) is the column vector that denotes the \( \ell_2 \) norm of each row of \( X \) and \( \Lambda \subseteq \{ 1, \cdots, K \} \) is a given index set. Ideally, if and only if \( \phi_j = \theta_k, X_{i}^{(j)} \neq 0 \), otherwise \( X_{i}^{(j)} = 0 \) [9].

As a result, the problem of DOA estimation with MMV can be changed to determining the index \( j \) in the set \( \Lambda \). Then, the joint-sparse recovery problem for DOA estimation can be described as [7-9]

\[
\min_{X} \| X \|_2 \quad \text{s.t.} \quad \| Y - \Phi X \|_F^2 \leq \beta^2, \quad (2)
\]

where \( \beta^2 \) is a regularization parameter, \( \| \cdot \|_F \) denotes Frobenius norm.

3. NOISE SUBSPACE WEIGHTED \( \ell_1 - \ell_2 \) ALGORITHM

The \( \ell_1 - \ell_2 \)-SVD algorithm enforces sparsity by the regular \( \ell_1 \) penalty [7-8]. Just as it is illustrated in [8, see Fig. 6.11 for details] and represented in [1], the regular \( \ell_1 \) minimization can not obtain exact recovery with a few snapshots. To deal with this problem, Candès et al designed an iterative reweighted formulation of \( \ell_1 \) minimization that large weights are appointed to the entries of the recovered signal whose indices are outside of the signal support [1].

The iterative \( \ell_1 \) reweighting is presented in [1]:

\[
w_{ij}^{(q+1)} = \left[ w_{ij}^{(q)} + \varepsilon \right]^{-1}, \quad (3)
\]

where \( x_i \) denotes the \( i \)-th entry of the recovered signal and \( w_{ij} \) is the corresponding weighted value, \( \varepsilon > 0 \) is an application-dependent parameter and it must be carefully designed, \( q \) is the iteration count number.

Here, we extend the idea of the iterative reweighted \( \ell_1 \) minimization from the SMV case to the MMV case. We exploit the orthogonality between the noise subspace and the array manifold matrix [5-6] to achieve this idea.

The correlation matrix for the measurements \( Y \) can be written as

\[
R = \mathbb{E} \left\{ y(t)y^H(t) \right\} = \mathbf{A} \mathbf{A}^H + \sigma^2 \mathbf{I}, \quad (4)
\]

where \( R = \mathbb{E} \left\{ s(t)s^H(t) \right\} \) denotes the correlation matrix of the signal, \( \mathbf{I} \in \mathbb{R}^{M \times M} \) is an identity matrix, \( (\cdot)^H \) denotes conjugate transpose.

By eigendecomposition, we have \( R = \sum_{m=1}^{M} \lambda_m u_m u_m^H \), where \( \lambda_m \) is the \( m \)-th eigenvalue and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M = \cdots = \lambda_0 \), \( u_m \) is the corresponding eigenvector. We define \( U = [u_1, \cdots, u_M] \) and \( U_s = [u_1, \cdots, u_s] \), which correspond to the signal and noise subspace, respectively. It was proved that [5-6]

\[
\mathbf{A}^H U_s = 0 \in \mathbb{R}^{p(M - s)}.
\]

Considering the relation between the overcomplete basis matrix \( \Phi \) and the array manifold matrix \( \mathbf{A} \), we can rewrite \( \Phi = [\mathbf{A}, \mathbf{B}] \), where \( \mathbf{B} \in \mathbb{C}^{M \times (K - p)} \). It is noted that the column index of \( \mathbf{A} \) and \( \mathbf{B} \) can be restricted to the inside and outside of the set \( \Lambda \), respectively.

Exploiting the property in (5), we have

\[
\Phi^H U_s = \begin{bmatrix} \mathbf{A}^H U_s \\ \mathbf{B}^H U_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{C} \end{bmatrix},
\]

where \( \mathbf{C}^{(i)} > 0 \) [6], \( \mathbf{C}^{(i)} \) denotes the \( i \)-th entry of \( \mathbf{C}^{(i)} \). In actual application, we have to substitute the sample correlation \( \hat{R} \) for \( R \), where \( \hat{R} = YY^H/T \). Consequently, we also have to substitute \( \hat{U}_s \) for \( U_s \). As a result, we can obtain the following equation

\[
\Phi^H \hat{U}_s = \begin{bmatrix} \mathbf{A}^H \hat{U}_s \\ \mathbf{B}^H \hat{U}_s \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \mathbf{W}.
\]

We can express the weighted vector as

\[
W^{(s)} = \begin{bmatrix} W_1^{(s)} \\ W_2^{(s)} \end{bmatrix}, \quad (8)
\]

When \( T \to \infty \), \( W_1^{(s)} \to 0^{(s)} \in \mathbb{R}^{p \times 1} \) and \( W_2^{(s)} \to \mathbf{C}^{(s)} \), and then the entries of \( W_2^{(s)} \) are smaller than those of \( W_1^{(s)} \).
We define
\[
Q = \text{diag}\left[ W^{(2)} \right],
\]  
(9)

\(Q\) is a diagonal matrix with \(W^{(2)}\) on the diagonal and zeros elsewhere. Consequently, we can employ \(Q\) as a weighted matrix to achieve the idea that the nonzero entries whose indices are inside of the row support of the jointly sparse signals are penalized by smaller weights and the other entries whose indices are more likely to be outside of the row support of the jointly sparse signals are penalized by larger weights.

Lastly, we can formulate the noise subspace weighted \(\ell_1\) minimization for sparse signal reconstruction:
\[
\min \|X\|_{Q;2,1} \quad \text{s.t.} \quad \|Y - \Phi X\| \leq \beta^2,
\]  
(10)

where \(\|X\|_{Q;2,1} = \sum \{Q \sum \|X_i\|^{1/2}\} [11]\).

Compared with the work of Candès et al [1], we obtain a proper weighted matrix for practical DOA estimation problem and it neither uses iterative process nor needs an application dependent parameter.

4. SIMULATIONS

Here, we present several simulations to demonstrate the performance of the proposed NSW- \(\ell_1 - \ell_2\) algorithm. We consider a uniform linear array (ULA) of \(M = 10\) sensors separated by half a wavelength in following simulations.

4.1 Spatial spectral of NSW- \(\ell_1 - \ell_2\)

We suppose that there are two uncorrelated signals impinging on the array from \(\theta_1 = -4^\circ\) and \(\theta_2 = 4^\circ\). The number of snapshots is 10, and the grid is uniform with 1° sampling from -90° to 90°, the SNR is 10dB. In Fig. 1, we compare the spectrum obtained using the proposed NSW-\(\ell_1 - \ell_2\) method with that of \(\ell_1\)-SVD. Although \(\ell_1\)-SVD can resolve the two sources, there are serious spurious peaks; while NSW- \(\ell_1 - \ell_2\) not only has more sharp peaks but also can suppress spurious peaks. In other words, by using the noise subspace weighted \(\ell_1\) minimization, sparsity can be enhanced, and then it can effectively suppress spurious peaks in the case of a few snapshots.

4.2 The accuracy of NSW- \(\ell_1 - \ell_2\)

We next compare the RMSE of the DOA estimates yielded by NSW- \(\ell_1 - \ell_2\) to that of \(\ell_1\)-SVD and to the CRB. In Fig. 2, we consider two uncorrelated sources at 0.15° and 23.32°. The idea of adaptively grid refining [7] is exploited to obtain RMSE. Since the weighted \(\ell_1\) minimization reduces the necessary sampling rate and enhance sparsity at the true source locations [1-4], the noise subspace weighted \(\ell_1\) minimization can accomplish accurate recovery by using a few snapshots. From the results in Fig. 2 it can be seen that the RMSE of NSW- \(\ell_1 - \ell_2\) estimates are closer to the CRB than that of \(\ell_1\)-SVD for uncorrelated sources.

4.3 Sensitivity of NSW- \(\ell_1 - \ell_2\) on subspace dimension determination

In general, the noise subspace dimension is usually unknown since the number of sources \(p\) is unknown. The number of sources can be decided from the measurements using the Minimum Description Length (MDL) [12]. The estimated number of sources from the MDL will become inaccurate, however, when the number of snapshots is small.
and SNR is low. Then we must consider the sensitivity of NSW-1A on subspace dimension determination. As can be seen from Fig.3, the variation in the spectra is slight for underestimated or overestimated the subspace dimension which does not incur catastrophic consequences for the sparse signal reconstruction. Hence the proposed NSW- $\ell_1 - \ell_2$ algorithm is not very sensitive on subspace dimension determination.

5. CONCLUSION

We proposed a DOA estimation algorithm based on weighted $\ell_1$ minimization that exploits the orthogonality between the noise subspace and the array manifold matrix to obtain the weighted values. The proposed algorithm penalizes nonzero entries whose indices are inside of the row support of the jointly sparse signals by smaller weights and the other entries whose indices are more likely to be outside of the row support of the jointly sparse signals by larger weights, and then it can enhance sparsity at the true source directions. Compared with the iterative reweighted $\ell_1$ approach from Candès et al, the proposed algorithm neither uses iterative process nor needs an application-dependent parameter. Numerical examples show that the proposed algorithm has better performance than $\ell_1$-SVD who exploited straightforwardly $\ell_1$ minimization. In addition, simulations also demonstrated that the proposed NSW- $\ell_1 - \ell_2$ algorithm is not very sensitive on subspace dimension (i.e. $M - p$) determination.

6. REFERENCES