DIRECTIONS-OF-ARRIVAL ESTIMATION USING A SPARSE SPATIAL SPECTRUM MODEL WITH UNCERTAINTY

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ABSTRACT
This paper is concerned with the estimation of the directions-of-arrival (DOA) of narrowband sources using a sparse spatial spectral model, with model uncertainty. When the model uncertainty is limited to DOAs not belonging to the search grid, off-grid error linearization given in [1], and constraint relaxation are used to effect a convex problem. This approach is useful either in reducing the dimension through the use of coarser candidate DOA grid, or improving the performance of estimators that use dense grids. In addition, a particular diagonal loading approach is proposed to reduce the sensitivity of the estimator to the choice of its regularization parameter.

Index Terms— Sparse Spectrum, Diagonal Loading

1. INTRODUCTION

Interest and developments in sparse representation and its application in signal recovery and parameter estimation have motivated several approaches for estimating the directions of arrival (DOA) of multiple sources by an array of sensors. Examples of early contributions to DOA estimation using sparsity are [2][3][4]. We call the method of [3] L1-SVD, which is formulated as a convex optimization problem and shows superior resolution performance than many traditional DOA estimation methods (e.g. MUSIC[5]). In [6], we proposed another approach, Sparse Spectral Fitting (SSF), which shares similar sparse signal representation ideas with L1-SVD. The SSF method shows comparable DOA estimation performance with L1-SVD, but appears more favorable in computational complexity and also directly estimates the signal powers. However, if the modeled steering vectors of these methods do not accurately approximate the true ones, their DOA estimation performance degrades and it becomes more difficult to select the method’s regularization parameters. In this paper, we propose a new convex DOA estimator, Sparse Spectral Fitting with Modeling Uncertainty (SSFMU), which deals with one of the main sources causing the inaccuracy of the steering vectors.

We begin with a summary of the sparse spectral representation and DOA estimator formulation based on second order statistics. Consider the classical narrow-band antenna array model of M elements with L arriving signals from directions $\theta_1, \theta_2, \ldots, \theta_L$. Assuming Additive White Gaussian Noise (AWGN), the spatial covariance matrix $R$ is defined by:

$$R = G S_x G^H + \sigma^2 I$$

where $G = [g(\theta_1), g(\theta_2), \ldots, g(\theta_L)] \in \mathbb{C}^{M \times L}$, with $g(\theta) \in \mathbb{C}^{M \times 1}$, denoting the steering vector at direction $\theta$; $H$ is conjugate transpose, $S_x \in \mathbb{C}^{L \times L}$, is the source covariance matrix and $\sigma^2$ is the noise power. In this paper, we only concern the very common case of uncorrelated zero-mean stationary sources and, thus, $S_x$ is real and diagonal.

Generally, the DOA estimation methods involve a search over a grid of candidate directions, which, in increasing order, are denoted by $\phi_1, \phi_2, \ldots, \phi_K$ and $K \gg L$ is the number of the directions of interest. If true DOAs belong to this grid, then the spatial covariance model (1.1) can be reformulated as a sparse representation:

$$R_v = A S + \sigma^2 I_v,$$

where $S = [s_1, s_2, \ldots, s_K]^T$ is the sparse spatial spectrum (since $K \gg L$) with $s_i = S_x(k, k)$ if $\exists k$ s.t. $\phi_i = \theta_k$, otherwise $s_i = 0$; $R_v = vec(R)$, $I_v = vec(I)$ and $A = [a(\phi_1), \ldots, a(\phi_K)]$ with $a(\phi) = vec(g(\phi)g^H(\phi))$. $A$ is called measurement matrix and $s_i$ represents the signal power from the direction bin around $\phi_i$. Now estimating the DOAs and signal powers from the under-determined linear equation system (1.2) reduces to a sparse reconstruction problem, which has been addressed in many contributions, e.g. [7].

In reality, the covariance matrix $R$ of (1.1) is not available and such a statistic must be estimated from data as, for example, in the form of the sample covariance matrix $C = \sum_{t=1}^{N-1} y(t)y^H(t)/N$, where $y(t)$, $M \times 1$, is the $t^{th}$ snapshot of the array output and $N$ is the number of the snapshots. Using an “error term” $e$ to summarize noise contribution, sample estimates and any other modeling errors in $C$, we have:

$$C_v = A S + e_v. \quad (1.3)$$

where $C_v = vec(C)$ and $e_v = vec(e)$. Utilizing the regularized recovery approach in [7], SSF is formulated as:

$$\min_{S} \|S\|_1 \quad s.t. \quad \|C_v - AS\|_2 \leq \beta \& S \geq 0,$$
where the last inequality is element-wise and $\beta$ is a regularization parameter which, by discrepancy principle, should satisfy $\beta \geq \|e_v\|_2$.

However, $g(\phi_i)$ may not accurately model the true steering vector of the signal from the DOA bin centered at $\phi_i$. This inaccuracy arises from many sources, e.g. calibration error and non-isotropic sensors. Among them, the failure of the assumption that $\{\theta_1, \ldots, \theta_K\} \subset \{\phi_1, \ldots, \phi_K\}$ is very common, since the candidate directions are discrete but the true DOAs vary continuously. If $g(\phi_i)$ is not accurate, $e_v$ in (1.3) may be enlarged in proportion to the signal strength, which may be a big problem for large SNR. According to the results of [7], this increase of $e_v$ generally degrades the DOA estimation performance of SSF and makes it harder to select $\beta$. This problem has been addressed in several contributions, e.g. [1][8]. The algorithms proposed in [8] are specifically for the case where $A$ is the discrete Fourier transformation matrix. In [1], a generally applicable method, Sparse Total Least Square (STLS), is proposed to estimate the candidate directions that are closest to the true DOAs along with the error in the modeled $A$. Unfortunately, STLS is non-convex.

In Section 2, we reformulate the problem, introduce an approximation and transform a non-convex estimator into a convex one, SSFMU, which incorporates the off-grid DOA $\theta \notin \{\phi_1, \ldots, \phi_K\}$ into the covariance matrix model (1.3). This is followed by Section 3 suggesting a diagonal loading approach to alleviate the sensitivity of SSFMU to its regularization parameter. Example statistics are provided in Section 4. Conclusions are given in Section 5.

2. SPARSE SPECTRAL FITTING WITH MODELING UNCERTAINTY

Suppose there is a signal from DOA $\theta_k$ which belongs to the DOA bin centered at $\phi_i$. We denote the true steering vector of this signal by $f(\theta_k)$ and $f(\theta_k) = vec(f(\theta_k)f^H(\theta_k))$. Then, we can have $f_i(\theta_k) = T^{(i)}a(\phi_i)$ where $T^{(i)}$, $M^2 \times M^2$, called correction matrix, is diagonal, complex and used to model the calibration error etc. Further, we can rewrite (1.3) as:

$$C_v = \sum_{i=1}^{K} s_i T^{(i)} a(\phi_i) + e_v,$$

where $e_v$ now does not contain any modeling error. Define matrix $T$ with its $i^{th}$ row $T_{i,:} = diag(T^{(i)})$. Observe that if $s_i = 0$ then $T_{i,:} = 0$, which suggests that $S$ and $T$ have the same sparsity pattern in row. Based on this observation, we take the group sparsity of $[S, T]$ as the objective function and formulate an estimator as:

$$\min_{S, T} \|S, T\|_{2,1} \text{ s.t. } \|C_v - \sum_{i=1}^{K} s_i T^{(i)} a(\phi_i)\|_2 \leq \beta$$

$$& s_i \geq 0, \quad i = 1, ..., K,$$

where

$$\|S, T\|_{2,1} = \sum_{i=1}^{K} \|s_i, T_{i,:}\|_2.$$  \hfill (2.2)

However, (2.2) is non-convex. To develop a convex estimator, in the following, we limit our concern to the modeling error introduced by the off-grid DOAs and transform (2.2) to a convex problem. For simplicity, we take Uniform Linear Array (ULA) as an example to present this transformation. For any other array geometry, it can be done in similar ways.

Now, if and only if we only consider the off-grid error in the model, we have $f_i(\theta_k) = a(\theta_k)$. For ULA, every entry in $a(\theta_k)$ has the form $e^{-j2\pi dm \sin \theta_i}$, where $d$ is the element spacing normalized by signal wavelength, $j = \sqrt{-1}$ and $m$ is an integer. We denote this entry by $a(\theta_k)_m$ and have:

$$a(\theta_k)_m = e^{-j2\pi dm \gamma_i} \times a(\phi_i)_{m},$$

where $e^{-j2\pi dm \gamma_i}$ is the corresponding entry of $T^{(i)}$ and $\gamma_i = \sin \theta_k - \sin \phi_i$ called correction parameter. By using Taylor expansion, we have $e^{-j2\pi dm \gamma_i} \approx 1 + (-j2\pi dm \gamma_i)$ and (2.3) can be re-written as:

$$a(\theta_k)_m \approx [1 + (-j2\pi dm \gamma_i)] a(\phi_i)_m.$$  \hfill (2.4)

Replacing $T^{(i)}a(\phi_i)$ by this approximation, (2.1) can be simplified to:

$$C_v = \sum_{i=1}^{K} [s_i a(\phi_i) + p_i b(\phi_i)] + e_v.$$  \hfill (2.5)

where $p_i = s_i \gamma_i$, $b(\phi_i) = E a(\phi_i)$ and $E$ is a diagonal matrix whose entry corresponding to $a(\phi)_m$ equals to $-j2\pi dm$. Since each $s_i$ represents the signal power from the corresponding direction bin, we let: $l_i \leq \gamma_i \leq u_i$, $i = 1, ..., K$, where:

$$l_i = \sin \frac{\phi_{i-1} + \phi_i}{2} - \sin \phi_i, \quad i = 2, ..., K,$$

with $l_1 = 0$ and

$$u_i = \sin \frac{\phi_{i+1} + \phi_i}{2} - \sin \phi_i, \quad i = 1, ..., K - 1,$$

with $u_K = 0$. Further, since $s_i \geq 0$ for $i = 1, ..., K$, we have:

$$s_i l_i \leq p_i \leq s_i u_i, \quad i = 1, ..., K.$$  \hfill (2.6)

Define $P = [p_1, p_2, ..., p_K]^T$. Similar as $[S, T]$, $S$ and $P$ have the same sparsity pattern. With this observation, the covariance matrix model (2.5) and the convex relaxation (2.6), we transform the non-convex optimization problem (2.2) into a convex one: Sparse Spectral Fitting with Modeling Uncertainty (SSFMU):

$$\min_{S, P} \|S, P\|_{2,1} \text{ s.t. } \|C_v - AS - BP\|_2 \leq \beta$$

$$s_i l_i \leq p_i \leq s_i u_i, \quad s_i \geq 0, \quad i = 1, ..., K,$$

(2.7)
where $B = [b(\phi_1), ..., b(\phi_K)]$ and $\beta$ is a regularization parameter. We denote the minimizer of (2.7) by $S^*$ and $P^*$, and the indices of the largest $L$ peaks in $S^*$ by $i_1, ..., i_L$. Then the DOA estimates are:

$$\hat{\theta}_k = \arcsin[\sin(\phi_k) + p_{i_k}^*/s_{i_k}^*], \quad k = 1, 2, ..., L, \quad (2.8)$$

where we assume $s_{i_k}^* > 0, k = 1, 2, ..., L$.

Obviously, SSFMU is a convex optimization problem, which can be solved very efficiently, and estimates the signal power along with the DOAs. It has two advantages over SSF, if using the same grid. We have also successfully applied similar diagonal loading approach to SSF.

### 3. DIAGONAL LOADING

Consider an equivalent formulation of (2.7):

$$\min_{S, P} \|C_v - AS - BP\|_2 + \delta \|S, P\|_{2,1} \quad (3.1)$$

$$s_i \leq p_i \leq s_i u_i, \quad s_i \geq 0, \quad i = 1, ..., K,$$

where $S$ and $P$ are the optimizers of (3.1) by $S^*$ and $P^*$. Our simulations show that, if $\delta = \sqrt{M}$, the fitting residual of (3.1), which is $C_v - AS^* - BP^*$, can be well approximated by $\alpha I_v$, where $\alpha$ is the size of the identity matrix components in the error term $e_\rho$ of (2.5) and $I_v = vec(I)$. Further, if we set $\delta = \sqrt{M} + \epsilon$ with $\epsilon$ being positive and sufficiently small, we have observed that the corresponding residual is approximately a linear function of $\alpha$:

$$\|C_v - AS^* - BP^*\|_2 \approx (\sqrt{M} + \alpha \epsilon) + c_1 \alpha + c_2, \quad (3.2)$$

where $c_1 > 0$ and $c_2$ are the coefficients. Especially, $c_1$ is very small and gets even smaller when $\epsilon$ becomes smaller. Note, for any given $\epsilon$, the equation (3.2) only holds for a finite range of $\alpha$. If $\alpha$ is too large, the optimizer $S^*$ and $P^*$ in (3.2) will be $0$. This is because:

$$\|C_v\|_2 \leq \|C_v - \alpha I_v\|_2 + \|\alpha I_v\|_2 = \|C_v - \alpha I_v\|_2 + \alpha \sqrt{M}.$$

If $\alpha$ is so large that $c_1 \alpha + c_2 \geq \|C_v - \alpha I_v\|_2$, which gives:

$$\|C_v - AS^* - BP^*\|_2 \geq \|C_v\|_2,$$

then by minimizing the objective function of (3.1), we get $S^* = P^* = 0$. Based on the above observations and analysis, we propose the following alternative formulation of (2.7):

$$\min_{S, P} \|C_v + \rho I_v - AS - BP\|_2 + \delta \|S, P\|_{2,1} \quad (3.3)$$

$$s_i \leq p_i \leq s_i u_i, \quad s_i \geq 0, \quad i = 1, ..., K,$$

where $\delta = \sqrt{M} + \epsilon$ with $\epsilon$ being positive and sufficiently small and $\rho$ is the parameter to be adjusted. According to (3.2), the 2-norm of the residual of (3.3) is approximately $(\sqrt{M} + c_1)(\rho + \alpha) + c_2$, where $(\rho + \alpha)\sqrt{M}$ accounts for $c_1 \rho + c_2$ for the other error terms in $e_\rho$. Since $c_1$ is very small, (3.3) will be much less sensitive to $\rho$ than (2.7) to $\beta$, which will be illustrated in Section 4. Furthermore, we can control this sensitivity by adjusting $\epsilon$.

Although there may be performance loss by using (3.3), its performance is at least suboptimal to (2.7) and better than SSF, if using the same grid. We have also successfully applied similar diagonal loading approach to SSF.

### 4. SIMULATION RESULTS

In this section, we first give an example illustrating the effectiveness of the diagonal loading approach. Then we present simulation results comparing the RMSEs of the DOA estimation of SSF, $L_1$-SVD and SSFMU.

We consider an ULA of $M=8$ sensors with $d=0.5$, receiving $L=2$ plane waves from $-54^\circ$ and $46^\circ$. These sources are assumed to be narrow-band, zero-mean and mutually uncorrelated. We set $N=3000$ and noise to be AWGN with unit variance. The candidate directional grids are assumed to be uniform and the DOA estimation error is defined as $\|\hat{\theta}_1 - \hat{\theta}_1, \hat{\theta}_2 - \hat{\theta}_2\|_2$.

In Fig. 1, two sources are of equal SNR=0dB and the grid is of $1^\circ$ separation with the grid points constituting the set $\{-90^\circ, -89^\circ, ..., 90^\circ\}$. For (3.3), we set $\delta = \sqrt{M} + 0.1$. Figure 1(a) and 1(b) show the curves of DOA estimation error versus $\beta$ of (2.7) and $\rho$ of (3.3), respectively. As shown in Fig. 1, when $2.95 < \beta < 4.75$, SSFMU (2.7) reaches its best (compared to the other values of $\beta$) performance and its DOA estimation Error is between 0.12 and 0.14, but when $\beta$ is outside this range, this error becomes larger than 0.57. For the diagonal loading formulation (3.3), its best performance is almost the same as that of SSFMU and attained in a much wider range $1.4 < \rho < 9$. Furthermore, when $\rho$ is outside this range but less than 24, its DOA estimation error is still less than 0.24. Through this example, it is illustrated that the diagonal loading approach can potentially keep the best performance of SSFMU while being much less sensitive to its regularization parameters.

In Fig. 2, we show a comparison of the RMSEs of the DOA estimations for SSF, $L_1$-SVD and SSFMU(2.7). We use 500 trials for each SNR and assume $L$ known to all the methods. The grids of SSF, $L_1$-SVD and SSFMU(dense) are the same as the grid used in Fig. 1, and the grid
of SSFMU(coarse) is of the same range but 4° separation. Therefore, the problem dimensions of $L_1$-SVD and SSFMU(dense) are the same, twice of that of SSF and four times of SSFMU(coarse). The regularization parameters of these methods are empirically and independently chosen. As shown in Fig. 2, all the four methods have the same large error threshold ($\approx -15$ dB) while SSFMU(coarse), with the lowest problem dimension, achieves similar or better performance, compared to SSF and $L_1$-SVD. Furthermore, SSFMU(dense), with the same dimension of $L_1$-SVD, attains much better DOA estimation performance than the other three especially when SNR is large. Note, when $L > 2$ the problem dimension of SSFMU(dense) is smaller than that of $L_1$-SVD and, in this example, when SNR $> 5$ dB the linear approximation error of SSFMU(coarse) becomes dominating and, thus, its performance becomes very similar to that of SSF and $L_1$-SVD.

5. CONCLUSIONS

In this paper, we presented a spatial spectrum estimator SSFMU which, by relaxing a more general non-convex estimator, was formulated as a convex optimization problem taking off-grid DOAs into account. Then, we proposed to use diagonal loading with a particular degree of loading to alleviate the sensitivity of SSFMU to its regularization parameter. Through simulations we showed the effectiveness of this diagonal loading method and compared the DOA estimation performance of SSFMU with SSF and $L_1$-SVD. The comparison demonstrated that SSFMU can either reduce the optimization problem dimension while keeping similar performances as SSF or achieve better DOA estimation performance with the same grid setting.

6. REFERENCES


