A novel method for estimating the direction of arrival of ultra-wideband wavefronts impinging on a linear uniform sensor array is proposed. It is based on expanding the space time signals onto a basis of Laguerre-Gauss Circular Harmonic functions, whose peculiar properties lead to a new signal subspace parametric model. This approach allows robust angle of arrival estimation in low signal to noise ratio environments.

Index Terms— UWB DOA, space-time array processing, Laguerre-Gauss Circular Harmonic expansion, UWB communications.

1. INTRODUCTION

Direction of arrival (DOA) estimation of plane waves is one of the most classical issues of sensor array signal processing, employed for localization of sources in propagating media. Several computationally efficient and accurate methods have been developed for narrow band sources [1] based on amplitude and phase differences of signals received by sensors. These classical methods are extended to the case of ultra-wideband (UWB) sources by focusing, i.e., by coherent combination of many narrowband components [1][2]. However, these techniques turn out to be computationally expensive and critical when the array bandwidth exceeds about one octave [3].

Another classical UWB DOA approach used in the specific case of a single radiating source is space-time (ST) processing, where the set of signals collected by a sensor array are viewed as 1-D patterns, i.e., 2-D signals constant in the direction orthogonal to the DOA. A well known ST estimation method consists on estimating the time difference of arrival (TDOA) between the signals received by any sensor pair [4], [5], [6]. TDOA estimates are prone to large outliers in low SNR environment.

The scope of this contribution is to present a method for estimating the DOA of UWB sources robust to high noise contamination. The method is based on expanding the ST signal impinging on a linear array onto a basis of Laguerre-Gauss Circular Harmonic (LG-CH) orthogonal functions [6][7]. These functions have two fundamental properties that make them suited for performing a kind of tomographic analysis of the ST signal, whose optimality for 1-D pattern orientation estimation was discussed in [8]. The first property is that they are polar separable and steer by multiplication by a complex factor [7]. The second one is that they are block-wise linearly related to corresponding sets of Cartesian separable two dimensional Hermite Gauss (2D-HG) orthogonal functions [9], [10], [11]. It is shown here that robust DOA estimates can be performed using a truncated series of coefficients [9], by means of polynomial rooting techniques.

2. NOTATION

Matrices are indicated by capital boldface letters, vectors by lowercase, boldface letters. The transpose of matrix $A$ is $A^T$. The Hermitian transpose of the same matrix is $A^H$. $I$ is the identity matrix. $E[\cdot]$ indicates the expected value.

3. SPACE TIME SIGNAL MODEL

A far-field source, located at angle $\theta$ with respect to the broadside of a uniform linear array (ULA) radiates the UWB signal $s(t)$ in a non-dispersive medium with wave propagation speed $c$. In a Cartesian reference system $(x,t)$, the real-valued baseband signal received by an unit gain, omni-directional sensor placed at the generic coordinates $(x,0)$ is given by
which defines a 2-D signal in the ST domain. ULA sensors are placed at positions \((q,0)\) for \(q=0,1,...,Q-1\) and their output signals are sampled at \(t=rT\), \(r=0,1,2,...,R-1\), where \(d\) is the normalized inter-sensor spacing, and \(T\) is the sampling period. Therefore the discrete ST signal is

\[
(, ) s i n () uqr s r T q d T . (3.2)
\]

After the substitutions

\[
1222cos( ) 1 sin ( ) d DT
\]

and

\[
1222sin( ) sin( ) 1 sin ( ) dd DT T
\]

(3.2) can be rewritten as

\[
(, ) cos( ) sin( )cos( ) Tuqr s r qDDD­½  ®¾ . (3.3)
\]

Let \(\text{max} f\) denote the upper frequency of the band-limited signal \((,) s t\), unambiguous sampling is obtained when

\[
2 \text{max} f d T 1 .
\]

In fact, \((,) uqr\) can be interpreted as a 1-D pattern \([6],[9]\) of a 2-D image on the \(qr\) plane, sampled at spatial frequency \(\cos(\alpha)/T\), which is rotated with respect to the horizontal axis by the angle \(\alpha=\arctan[d\sin(\theta)]\). See Fig. 1. Thus, the problem of estimating the UWB signal DOA \(\theta\) reduces to estimating the orientation \(\alpha\) of the underlying 1-D pattern \([9]\).

4. LG-CH SIGNAL SUBSPACE

It is assumed that \(R\) consecutive baseband UWB snapshots, band-limited at frequency \((2T)^{-1}\) and sampled with period \(T\), are collected from a \(Q\)-sensor ULA, forming a patch of the 1-D pattern. Considering the coordinates \(q\) and \(r\) as continuous variables, the LG-CH expansion of the patch

\[
I(q,r) = u\left(q - \frac{Q-1}{2}, r - \frac{R-1}{2}\right) (4.1)
\]

written in polar coordinates \(\rho = \sqrt{q^2 + r^2}\) and \(\gamma = \arctan(r/q)\) is expressed at scale \(\sigma\) as

\[
I(\rho,\gamma) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} l_{n,k} L_n^m(\rho,\gamma;\sigma). (4.2)
\]

\(L_n^m(\rho,\gamma;\sigma)\) are the orthonormal, polar separable LG-CH functions

\[
L_n^m(\rho,\gamma;\sigma) = \frac{1}{\sqrt{k!|k|!}\sigma^{1/2}} e^{-\frac{1}{2}\sigma}(\rho/\sigma)^m e^{in\gamma} . (4.3)
\]

where \(L_n^k(\cdot)\) is a generalized Laguerre polynomial \([9]\). The index \(k\) is referred to as the radial order and the index \(n\) as the angular order of the LG-CH function.

The expansion coefficients \(l_{n,k}\) in (4.2), that in continuous and unbounded space-time are the scalar products of \((,) uxt\) with the LG-CH functions (4.3), can be calculated from the \(Q\timesR\) samples of the patch \(I(q,r)\) by over-determined Least Squares fitting \([11],[12]\). The expansion (4.2) is truncated to a finite order \(M\) by retaining the specific subsets \(\{l_{m-k,n}\}\) for \(k=0,1,...,m\) of LG-CH coefficients for \(m=0,1,...,M\) \([9],[11]\), each grouped in a row vector \(g_m\) of length \((m+1)\). A LG-CH column snapshot of length \((M+1)(M+2)/2\) is finally constructed as

\[
g = [g_0, g_1, ..., g_M]^T . (4.4)
\]

The choices of the \(M\) and \(\sigma\) depend on the array size \(Q\). In particular, to minimize the loss of orthogonality of LG-CH sampled functions within the patch \([10],[9]\), \(\sigma\) and \(M\) should satisfy

\[
\sigma < \min(R,Q) \frac{1}{2\sqrt{4M+1}} . (4.5)
\]

while the LG-CH expansion bandwidth, proportional to \((2\pi\sigma)^{-1}\sqrt{4M+1}\) \([10]\), should in turn be smaller than...
(2T)^{-1} to avoid aliasing problems. See [9] for a table of optimized \( \sigma \) values for practical orders. The first fundamental property of the LG-CH functions for the problem of DOA estimation is that they are polar separable and steer to an arbitrary orientation \( \alpha \) by simple multiplication by the phase factor \( e^{j\alpha} \) [6]. The second fundamental property is that they span the same signal space as the 2-D HG functions [9], [10], [11], [12]. More specifically, the components of each \( \mathbf{g}_m \) are linearly related as

\[ \mathbf{g}_m = \mathbf{T}_m \mathbf{f}_m \]

to the vector \( \mathbf{f}_m \) containing the subset \( \{ f_{m-1,l}, l = 0,1,\ldots,M \} \) of the 2-D HG expansion coefficients defined by [9]

\[ I(x,t) = \sum_{m=0}^{M} \sum_{l=0}^{M} f_{m-1,l} \phi_{m-1,l}(x,t;\sigma) \]  

(4.6)

where

\[ \phi_{m-1,l}(x,t;\sigma) = \frac{H_{m-1}(\frac{x}{\sigma})}{\sqrt{2^{m-1}(m-1)!\sigma\sqrt{\pi}}} \times \frac{H_{l}(\frac{t}{\sigma})}{\sqrt{2^l l!\sigma\sqrt{\pi}}} e^{\frac{x^2}{2\sigma^2}} \]

is the 2-D HG orthonormal function of Cartesian order \((m-1,l)\) [11] and \( H_l(x) \) is the Hermite polynomial of order \( l \) [13].

Based on these relationships, in presence of additive noise (4.4) it can be shown that the coefficients of the LG-CH expansion (4.4) obey [9]

\[ \mathbf{g} = \mathbf{D}(\alpha) \mathbf{E}_v \mathbf{c} + \mathbf{v} \]  

(4.7)

where \( \mathbf{D}(\alpha) \) is a rotation (unitary) diagonal matrix with non-zero entries \( e^{j\alpha m} \), \( \mathbf{E}_v \) is a complex-valued, orthogonal matrix of size \( (M+1)(M+2)/2 \times (M+1) \) [9] and \( \mathbf{v} \) is the random vector of the observation noise in the LG-CH expansion; \( \mathbf{c} \) is a real-valued vector of size \( (M+1) \times 1 \), whose components satisfy

\[ c_m = \left( \sum_{k=0}^{M} \frac{(-1)^k \sqrt{\pi} H_k(0)}{2^{k+1} k!} \right) b_m \]  

(4.8)

for \( m = 0,1,\ldots,M \), where

\[ b_m = \int_{-\infty}^{+\infty} \frac{t \cos(\alpha)}{\cos(\alpha)} \frac{H_m\left(\frac{t}{\sigma}\right)}{\sqrt{2^{m+1} m!\sigma\sqrt{\pi}}} e^{\frac{t^2}{2\sigma^2}} dt \]  

(4.9)

is the \( m \)th order 1-D Hermite-Gauss expansion [11] coefficient of the time-stretched copy of the source signal \( \sqrt{\frac{t}{\cos(\alpha)}} \). It is recognized that model (4.7) is a kind of tomographic representation of the ST signal (3.1). In fact (4.7) amounts to say that the truncated LG-CH expansion of the patch is the rotated tomographic back-projection of an 1-D pattern described by the coefficients \( \mathbf{c} \), that are scaled versions of the coefficients of the 1-D Hermite expansion of the stretched source signal. Back-projection is performed by the matrix \( \mathbf{E}_v \), while rotation by the angle \( \alpha \) is performed by the steering matrix \( \mathbf{D}(\alpha) \). This tomographic model naturally extends to multiple sources and can be used for general array processing of UWB waves, in some analogy with conventional narrowband models.

### 5. THE LG-CH CML ESTIMATOR

Under the hypothesis that \( \mathbf{v} \) represents a zero-mean, Gaussian noise with covariance \( \mathbf{E}\left[ \mathbf{vv}^{H} \right] = \mathbf{R}_v = \mathbf{C}_v \mathbf{C}_v^{H} \), the conditional maximum likelihood (CML) estimator of \( \alpha \) is obtained by solving the separable LS optimization problem

\[ \hat{\alpha}_{CML} = \arg \min_{\alpha, \mathbf{c}} \| \mathbf{c} - \mathbf{D}(\alpha) \mathbf{E}_v \mathbf{c} \|_2 \]  

(5.1)

where

\[ \mathbf{c}(\alpha) = \left[ \mathbf{E}_v^{H} \mathbf{D}(\alpha) \mathbf{R}_v \mathbf{D}(\alpha) \mathbf{E}_v \right]^{-1} \mathbf{E}_v^{H} \mathbf{D}(\alpha) \mathbf{R}_v \mathbf{g} \]  

(5.2)

is the conditional estimate of \( \mathbf{c} \) given \( \alpha \) [1], [9]. For \( M \rightarrow \infty \), which implies \( R, Q, \sigma \rightarrow \infty \), and assuming perfect LG-CH and array modeling, the CML estimate is asymptotically efficient [9] with respect to the Cramer-Rao bound (CRB) of the LG-CH beamspace [2], [3], which is in turn lower-bounded by the intrinsic ULA CRB [3]. The CML estimate \( \hat{\mathbf{c}}_{CML} \) of \( \mathbf{c} \) (and therefore of \( \{ b_m; m = 0,1,\ldots,M \} \) through (4.8)) is given by [9]

\[ \hat{\mathbf{c}}_{CML} = \mathbf{c}(\hat{\alpha}_{CML}) \]  

(5.3)

If \( \mathbf{D}(\alpha) \mathbf{R}_v \mathbf{D}(\alpha) = \mathbf{R}_v \) (e.g., in the common case when \( \mathbf{R}_v = \mathbf{I} \) or diagonal), the optimization can be quickly carried out by polynomial rooting after setting \( z = e^{j\alpha} \) [9]. In essence, the CML estimate (5.1) consists of finding the angle \( \alpha \) so that the weighted difference between the measured coefficients \( \mathbf{g} \) and their noise free tomographic model has minimum energy, ideally corresponding to the additive noise energy. The algorithm first calculates the LG-CH expansion coefficients, forms the polynomial expressed by the argument of (5.1) by the substitution \( z = e^{j\alpha} \) and finally performs polynomial rooting. The calculus of the LG-CH coefficients is performed through straightforward LS fitting on ST raw data (4.1). This is the most demanding task of the LG-CH CML estimate from a computational viewpoint.

### 6. COMPUTER SIMULATIONS

To put into evidence the features of the LG-CH CML estimator, an ULA of \( Q = 14 \) sensors with normalized intersensor distance \( d = 0.9 \) was selected for the simulation of single source UWB DOA estimation.
A unit power wavefront coming from $\theta = 35$ degrees from broadside and affected by additive white Gaussian noise was considered. The signal-to-noise ratio (SNR) refers to a single sensor.

Since the LG-CH expansion fits the ST signal within a circle of approximate radius $\sigma \sqrt{4 M + 1}$ [10], the CML estimate of order $M = 8$ and $\sigma = 1.46 T$ was performed on a series of 8 consecutive, partially overlapped patches, each of time length $R = 14$.

The results of the simulation are compared for benchmarking purposes with the results of the state of art TDOA algorithm described in [6], adapted to single source DOA estimation with ULAs. The TDOAs were estimated by the squared AMDF algorithm [5] applied to the whole data set employed for LG-CH estimation using parabolic interpolation after eight times oversampling of the sensor signals. Thecomputational burden of the TDOA is concentrated on the estimation of the single relative delays between sensor pairs and is comparable with the one of the LG-CH CML in this experiment.

Results are reported in Fig. 2. As expected, TDOA based estimates are more accurate with respect to LG-CH CML estimates at medium-high SNR, since they exploit the whole information contained in the ST support, while LG-CH CML estimates are based on subspace representations. On the contrary, TDOA estimates do not possess the tomographic projection features which makes the LG-CH CML estimates more robust in high noise environments.

### 7. CONCLUDING REMARKS

Based on the special properties of the LG-CH expansion a novel DOA estimator has been proposed. The LG-CH CML method presented here is not only interesting for its remarkable robustness to noise. In fact, it provides simultaneous accurate extraction of the Hermite coefficients of the source signal. In this regard, it is worth to point out that pulsed signals used in UWB communications are often modeled by low order Hermite-Gauss expansions [14].

From a wider perspective, the LG-CH approach opens the way to flexible UWB array processing techniques based on tomographic concepts and circular harmonic functions. These functions are steerable by simple multiplication by phase factors, in some analogy to steering made by classical approaches for narrow band sources. Extension of this approach to adaptive beamforming and multi-source DOA estimation are the matter of current developments.

### 11. REFERENCES


