ABSTRACT

This work deals with the Sequential Cramér-Rao Lower Bound (SCRLB) for sequential target state estimators for a bistatic tracking problem. In the context of tracking, the SCRLB provides a powerful tool, enabling one to determine a lower bound on the optimal achievable accuracy of target state estimation. The bistatic SCRLBs are analyzed and compared to the monostatic counterparts for a fixed target trajectory. Two different kinematic models are analyzed: constant velocity and constant acceleration. The derived bounds are also valid when the target trajectory is characterized by the combination of these two motions.

Index Terms— Sequential Cramér-Rao Lower Bound, Fisher Information, Bistatic Radar, Radar Tracking.

1. INTRODUCTION

A bistatic radar is a system in which transmitter and receiver are at separate locations. In the last years, as proved by the numerous experimental systems being built and the results reported in the literature, there is a great interest in these systems. Bistatic radars are very interesting because they can operate with their own dedicated transmitters, designed for bistatic operation, or with transmitters of opportunity, which are designed for other purposes but suitable for bistatic operation. In previous works [4],[5],[6], we evaluated the bistatic Cramér-Rao Lower Bound (CRLB) for the target range and velocity both for active and passive systems. In particular, the performance in estimating these two parameters, considered here as the radar measurements, strongly depends on the bistatic geometry which clearly changes while the target is moving along its trajectory.

In this work, exploiting the general method provided in [1], we derive the SCRLB of target state for bistatic tracking. The definitions of CRLB and SCRLB are similar. The CRLB is defined to be the inverse of the Fisher Information matrix and provides a mean square error bound on the performance of any unbiased estimator of an unknown parameter vector. The bound is referred to as the SCRLB if this parameter vector is sequentially estimated and there is an information gain given by the previous estimates. The problem of developing the SCRLB for bistatic tracking has been analyzed in literature, anyway the covariance matrix of the measurements has been always modeled constant and independent of the geometry [2].

The novel contribution of this work is that here the SCRLB of the target state is derived considering the covariance matrix of the radar measurements dependent on the target trajectory. The bistatic bounds derived in this work are also valid for monostatic radar, considered as a bistatic system where the distance from transmitter to receiver is null. The SCRLBs of both systems are then analyzed and compared.

2. ANALYZED SCENARIO

The geometry of the analyzed scenario is two-dimensional, as showed in Figure 1, where the receiver is located at the origin while the transmitter is on the y axis at a distance from the receiver equal to the baseline L.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tx=(0,L)</td>
<td>Tg</td>
</tr>
<tr>
<td>Rx=(0,0)</td>
<td></td>
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</tbody>
</table>

The target is moving with the trajectory showed in Figure 1. Measuring the target delay and the Doppler shift, the receiver is able to evaluate the range from the receiver to target and the bistatic velocity, that is, the component of the target velocity in the direction of the bisector of the angle at the vertex which represents the target [4]. The receiver look angle is assumed known. The basic problem is to estimate the target position and velocity from noise corrupted range and bistatic velocity data. Next we define the problem mathematically by considering two target motion models, the constant velocity and the constant acceleration, that is, the component of the target velocity in the direction of the bisector of the angle at the vertex which represents the target [4]. The receiver look angle is assumed known. The basic problem is to estimate the target position and velocity from noise corrupted range and bistatic velocity data. Next we define the problem mathematically by considering two target motion models, the constant velocity and the constant acceleration motions.

Then, we derive the CRLB for sequential estimators of the target state for the zero process noise case, that is, when the target trajectory is purely deterministic.

The SCRLB are derived assuming that the measurement sensor is operating with a unitary detection probability.
2.1 Constant velocity motion

Let’s consider first the problem of the constant velocity motion. In this case the target, located at \((x, y)\), is assumed to move with a constant velocity \((\dot{x}, \dot{y})\), with a state vector defined as \(\mathbf{x} = [x, \dot{x}, y, \dot{y}]^T\). Assuming that the evolution of the state vector is purely deterministic, it is possible to write \(\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k\), where:

\[
\mathbf{F} = \begin{bmatrix}
1 & T & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & T \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(1)

and \(T\) is the sampling time.

The available measurements at time \(k\) are the range from receiver to target and the bistatic velocity. The measurement equations can be put in the following vectorial form:

\[
\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{w}_k
\]

(2)

where \(\mathbf{z}_k\) is the collection of the bistatic measurements at the \(k\)th time instant while \(\mathbf{h}(\mathbf{x}_k) = [r_k, v_x, v_y]^T = [h_1(\mathbf{x}_k), h_2(\mathbf{x}_k)]^T\) is a non-linear vector function of the state vector \(\mathbf{x}_k\). The bistatic measurements are affected by additive Gaussian noise \(\mathbf{w}_k\) with zero mean and covariance matrix \(\mathbf{R}_k\). To give explicit expression of \(\mathbf{h}(\mathbf{x}_k)\), referring to the bistatic geometry of Figure 1, it is easy to verify that

\[
h_1(\mathbf{x}_k) = r_k = \sqrt{x_k^2 + y_k^2}, \quad h_2(\mathbf{x}_k) = v_k = \frac{x_k \dot{x}_k + y_k \dot{y}_k}{\sqrt{x_k^2 + y_k^2}}
\]

(3)

where

\[
x_k = \frac{L}{L + r_k + r_t} x_k - x_k, \quad y_k = \frac{L}{L + r_k + r_t} (y_k + r_k) - y_k
\]

(4)

\[
r_k = \sqrt{x_k^2 + y_k^2}
\]

(5)

The problem of developing the SCRLB for bistatic radar tracking has been analyzed in [2] using different target measurements, anyway the matrix \(\mathbf{R}_k\) has been modeled constant and independent of the bistatic geometry.

As known (see [4] and [6]), in bistatic radar systems, the performance in estimating the range and the bistatic velocity heavily depends on the transmitted waveform and on the geometry of the scenario, that is, the position of receivers and transmitters with respect to the position of the target.

In [4] and [6], the authors showed that the Fisher Information Matrix (FIM) of the range and the bistatic velocity is \(\mathbf{J}_K = \mathbf{P}_K^{1/2} \mathbf{J}_M \mathbf{P}_K^{1/2}\) where, \(\mathbf{J}_M\) is constant and depends only on the transmitted waveform, while \(\mathbf{P}_K\) depends on the geometry. In particular, \(\mathbf{P}_K\) is defined as

\[
\mathbf{P}_K = \begin{bmatrix}
\frac{\partial r_k}{\partial r_k} & \frac{\partial r_k}{\partial \dot{r}_k} \\
\frac{\partial v_k}{\partial r_k} & \frac{\partial v_k}{\partial \dot{r}_k}
\end{bmatrix}
\]

(6)

where \(r_k\) and \(\dot{r}_k\) are the delay and the Doppler shift of the radar target, that, referring to Figure 1, should be obtained using the geometry dependent non linear equations:

\[
r_k = r_0 + \sqrt{r_0^2 + L^2 + 2r_0 L \sin \theta_k},
\]

(7)

\[
\dot{r}_k = \frac{2 f_c v_k}{c} \sqrt{\frac{1}{2} \left[\frac{r_0 + L \sin \theta_k}{\sqrt{r_0^2 + L^2 + 2r_0 L \sin \theta_k}} - \frac{r_0 + L \sin \theta_k}{2} \right]},
\]

(8)

where \(c\) is the speed of light, \(f_c\) is the carrier frequency and \(\theta_k\) is the receiver look angle.

Matrix \(\mathbf{J}_M\) depends on the transmitted waveform. When the transmitted signal is a sequence of linear frequency modulated (LFM) pulses (chirps), \(\mathbf{J}_M\) is given by [4], [6]:

\[
\mathbf{J}_M = -\frac{2 T_p^2 \pi^2 \text{SNR}}{3} \begin{bmatrix}
-\beta^2 & \beta \\
\beta & -1 - \left(\frac{T_R}{T_p}\right)^2 \left(N_s^2 - 1\right)
\end{bmatrix}
\]

(9)

where \(N_s\) is the number of pulses of the transmitted burst, \(T_R\) is the pulse repetition time and \(T_p\) is the duration of each pulse, with \(T_p < T_R/2\). Moreover, \(\beta T_p^2 = B T_p\) is the signal effective time-bandwidth product and \(B\) is the total frequency deviation. The signal-to-noise ratio (SNR) is inversely proportional to the path loss factor \((T_d r_t)^\gamma\) due to propagation.

Assuming that the receiver is efficient in estimating the range and the bistatic velocity, the inverse of the covariance matrix \(\mathbf{R}_k\) is given by

\[
\mathbf{R}_k^{-1} = \mathbf{P}_K^{1/2} \mathbf{J}_M^{-1} \mathbf{P}_K^{1/2},
\]

(10)

where \(\mathbf{H}_k\) is the Jacobian of \(\mathbf{h}(\mathbf{x}_{k+1})\) evaluated at the true state \(\mathbf{x}_{k+1}\); that is:

\[
\mathbf{H}_k = \begin{bmatrix}
\frac{\partial h_1}{\partial x_{k+1}} & \frac{\partial h_1}{\partial \dot{x}_{k+1}} & \frac{\partial h_1}{\partial y_{k+1}} & \frac{\partial h_1}{\partial \dot{y}_{k+1}} \\
\frac{\partial h_2}{\partial x_{k+1}} & \frac{\partial h_2}{\partial \dot{x}_{k+1}} & \frac{\partial h_2}{\partial y_{k+1}} & \frac{\partial h_2}{\partial \dot{y}_{k+1}}
\end{bmatrix}
\]

(11)

The expressions of the elements of \(\mathbf{H}_k\) are given by

\[
\frac{\partial h_1}{\partial x_{k+1}} = \frac{x_{k+1}}{r_{k+1}}, \quad \frac{\partial h_1}{\partial \dot{x}_{k+1}} = 0,
\]

(12)

\[
\frac{\partial h_1}{\partial y_{k+1}} = \frac{y_{k+1}}{r_{k+1}}, \quad \frac{\partial h_1}{\partial \dot{y}_{k+1}} = 0,
\]

(13)
Considering the zero process noise case, the FIM of the state vector $\mathbf{x}_k$ can be obtained using the following equation:

$$
\mathbf{J}_k = \left[ \mathbf{F}_k^{-1} \right]^T \mathbf{J}_k \mathbf{F}_k^{-1} + \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k,
$$

where

$$
\mathbf{F}_k^{-1} = \begin{bmatrix} \mathbf{F} & \mathbf{G}_k \end{bmatrix} \mathbf{I}_2.
$$

and, considering that the measurements of the range and the bistatic velocity do not depend on the acceleration,

$$
\mathbf{H}_k = \begin{bmatrix} \mathbf{H}_k & \mathbf{0}_{2 \times 2} \end{bmatrix}.
$$

In this work, we are interested on evaluating the SCRLB of the target position and velocity. Therefore, it is useful to write the FIM in (19) using the following block matrix notation:

$$
\mathbf{J}_k = \begin{bmatrix} \mathbf{J}_{k+1} & \mathbf{J}_{k+1} \end{bmatrix},
$$

where the top-left matrix $\mathbf{J}_k$ is the $4 \times 4$ information matrix of the state vector $\mathbf{x}_k$. After straightforward manipulation, it is easy to verify that:

$$
\mathbf{J}_{k+1} = \left[ \mathbf{F}_k^{-1} \right]^T \mathbf{J}_k \mathbf{F}_k^{-1} + \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k + \mathbf{G}_k^T \mathbf{F}_k^{-1} \mathbf{J}_k \mathbf{F}_k^{-1} \mathbf{G}_k + \mathbf{G}_k^T \mathbf{F}_k^{-1} \mathbf{J}_k + \mathbf{J}_k.
$$

The SCRLBs on target position and target velocity estimation accuracy are given by the first four element on the principal diagonal of the inverse of $\mathbf{J}_k$. In particular, the top-left $4 \times 4$ sub-matrix of the inverse of $\mathbf{J}_k$ can be obtained as:

$$
\mathbf{J}_{k+1}^{-1} = \left( \mathbf{J}_{k+1} - \mathbf{J}_{k+1} \mathbf{J}_{k+1}^T \right)^{-1}.
$$

With respect to (10), in eq. (26) there is a term that takes into account the effects of acceleration.

Moreover, considering $\mathbf{G}_k = \mathbf{0}_{2 \times 2}$, that is forcing the acceleration to zero, it is easy to verify that (26) is similar to (10), the only difference is a term that takes into account the loss of information due to the estimation of the target acceleration performed with the same dataset. Even in this case, the monostatic SCRLBs are given by forcing the baseline $L=0$. It is clear that the result in (10) is useful only when the target is moving with constant velocity.

On the other hand, when the target trajectory is characterized by the combination of different target motions, i.e. constant velocity followed by constant acceleration, the useful equation is (26) obtained by switching properly the matrix $\mathbf{G}_k$. 

\[\text{(14)}\]
\[
\frac{\partial h}{\partial x_{k+i}} = \frac{1}{d_{k+i}^2} \left[ \left( \frac{\partial x_{k+i}}{\partial x_{k+i}} + \frac{\partial y_{k+i}}{\partial y_{k+i}} \right) d_{k+i} + \cdots \right],
\]
\[
= \frac{1}{d_{k+i}^2} \left[ \left( \frac{\partial x_{k+i}}{\partial x_{k+i}} + \frac{\partial y_{k+i}}{\partial y_{k+i}} \right) d_{k+i} + \cdots \right],
\]
\[\text{(15)}\]
\[
\frac{\partial h}{\partial y_{k+i}} = \frac{1}{d_{k+i}^2} \left[ \left( \frac{\partial x_{k+i}}{\partial y_{k+i}} + \frac{\partial y_{k+i}}{\partial y_{k+i}} \right) d_{k+i} + \cdots \right],
\]
\[\text{(16)}\]
\[
\frac{\partial h}{\partial x_{k+i}} = \frac{\tilde{x}_{k+i}}{d_{k+i}}, \quad \frac{\partial h}{\partial y_{k+i}} = \frac{\tilde{y}_{k+i}}{d_{k+i}},
\]
\[
d_{k+i} = \sqrt{\tilde{x}_{k+i}^2 + \tilde{y}_{k+i}^2}.
\]

The derivatives that appear in (14) and (15) can be straightforwardly derived and are not reported here for lack of space. From these equations it is clear that the FIM $\mathbf{J}_k$ depends on target trajectory, sensor accuracy (through $\mathbf{R}_k$, which is itself dependent on the target trajectory and on the transmitted waveform), the sampling interval $T$ and the baseline length $L$. In particular, it is known that when the target is crossing the baseline, resolution is totally lost and therefore the errors of the measurements tends to infinity [4]. Matrix $\mathbf{R}_k^{-1}$ tends to zero and hence also the second term in (10). In this case there is no information gain collecting the target measurements. Moreover, it is clear that for $L=0$ the bistatic SCRLBs coincide with the monostatic SCRLBs (the SCRLB are given by taking the diagonal elements of the inverse of $\mathbf{J}_k$).

In the monostatic case, matrix $\mathbf{P}_0$ of (6) becomes diagonal and constant, therefore the errors on the measurements become independent of the geometry. Concluding, the recursion in (10) starts with the initial FIM $\mathbf{J}_0$ that, assuming the initial distribution of $\mathbf{x}_0$ is Gaussian, is equal to the inverse of the covariance matrix of $\mathbf{x}_0$ [1], [2].

### 2.2 Constant acceleration motion

For the constant acceleration motion the problems is very similar to the one treated in the previous subsection. The state vector is defined as $\mathbf{x} = [\mathbf{x}, \mathbf{a}]^T = [x, \dot{x}, y, \dot{y}, \ddot{x}, \ddot{y}, \dddot{x}, \dddot{y}]^T$, where $\dddot{x}$ and $\dddot{y}$ are the accelerations along the $x$ and $y$ directions, which are assumed constant. Even in this case the state equation is linear and is defined as $\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k$, where

$$
\mathbf{F} = \begin{bmatrix} \mathbf{F} & \mathbf{G}_k \end{bmatrix} \mathbf{I}_2.
$$

\[\text{(17)}\]

$\mathbf{F}$ is the same matrix defined in (1), $\mathbf{I}_2$ is the $2 \times 2$ identity matrix, $\mathbf{0}_{2 \times 2}$ is a matrix whose elements are zero while $\mathbf{G}_k$ is a matrix which takes into account the effect of acceleration. In particular, $\mathbf{G}_k = \mathbf{0}_{2 \times 2}$ for the constant velocity motion, while $\mathbf{G}_k = \mathbf{G}$ for the constant acceleration motion, where

$$
\mathbf{G} = \begin{bmatrix} \frac{T^2}{2} & T & 0 & 0 \\
0 & 0 & \frac{T^2}{2} & T \end{bmatrix}.
$$

\[\text{(18)}\]
3. RESULTS & CONCLUSIONS

The SCRLBs of target position and velocity has been evaluated referring to the scenario of Fig. 1. We considered both the monostatic and the bistatic cases. In the bistatic configuration, the baseline has been fixed to 50Km. The transmitted signal is a burst of \( N_p = 8 \) LFM pulses, where \( T_p = 250 \mu \text{sec}, \ T_R = 1 \text{msec}, \ B = 1 \text{MHz}, \ f_c = 10 \text{GHz} \). The target trajectory is formed of two straight lines separated by a centripetal acceleration motion. On the straight lines, the target moves with constant velocity. At \( t = 0 \) the target is in \( (x=10L, \ y=L/2) \) moving with constant velocity \( (\dot{x} = -100, \ \dot{y} = 0) \) m/sec. At \( t=5000\text{sec} \) the target crosses the baseline while at \( t=5400\text{sec} \) it starts the manoeuvre whose duration is 360 sec. The trajectory is sampled with a sampling interval of \( T = 1\text{sec} \). Using the results obtained in Sect. 2.2, Figs. 2-5 show the SCRLBs of the target position and target velocity for the monostatic radar (in red, \( L=0 \)) and for the bistatic case (in blue, \( L=50 \text{ km} \)). Since the magnitudes of errors, as predicted by the SCRLBs, are very large in the initial interval, the bounds are plotted for \( t>4000\text{sec} \). In order to highlight the dependence of the performance of bistatic system on the geometry, the bounds have been calculated by keeping constant the \( \text{SNR} \) at 0dB. Initially, the performance of the monostatic system and the bistatic one are the same. When the distance from receiver to target is one order of magnitude greater than the baseline, the bistatic system behaves as the monostatic one. On the other hand, when the target approaches the baseline, the information due to the target measurements tends to zero and the information gain is only due to the a priori information.

4. REFERENCES


