ROBUST ADAPTIVE BEAMFORMING BASED ON JOINTLY ESTIMATING COVARIANCE MATRIX AND STEERING VECTOR

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ABSTRACT

In this paper, a new adaptive beamforming algorithm with joint robustness against covariance matrix uncertainty as well as steering vector mismatch is proposed. First, the theoretical covariance matrix is estimated based on the shrinkage method. Subsequently, the difference between the actual and the presumed steering vector is estimated by solving a quadratic convex optimization problem, which enables correction of the presumed steering vector. Unlike other robust beamforming techniques, neither the norm of the steering vector nor the upper bound of the norm of the mismatch vector is assumed in our approach. Simulation results show the effectiveness of the proposed algorithm both in terms of output performance and computational complexity.

Index Terms— Adaptive beamforming, quadratic programming, robustness, shrinkage estimation

1. INTRODUCTION

In the past three decades, many techniques have been proposed to improve the robustness of adaptive beamformers. In general, they can be classified into two categories based on the fundamental Capon beamformer. One is to process the sample covariance matrix, such as diagonal loading [1] and eigen-decomposition [2]. The other is to process the desired signal steering vector, such as worst-case performance optimization [3], equivalent directions of arrival (DOAs) estimation [4] and mismatch vector estimation [5]. However, these two categories of robust adaptive beamforming techniques were developed almost independently. Even though there are some works dealing with joint robustness against covariance matrix uncertainty and steering vector mismatch, many strong assumptions have been made in these methods, such as the norm of mismatch steering vector being upper-bounded [6]. Unfortunately, neither the mismatch steering vector nor its upper bound is a priori known in practical applications. Therefore, these adaptive beamforming techniques are not optimal, since they are based on an assumption that either the covariance matrix or the steering vector is exactly known. In [7], the authors paid equal attention to the two factors of the Capon beamformer simultaneously rather than just one of them, and achieved better performance. However, the norm constraint of the steering vector is still impractical in real applications. In addition, the computational complexity is relatively high because iteration is usually unavoidable especially at low SNR.

In this paper, we propose a new robust adaptive beamforming algorithm, which is based on estimating the theoretical covariance matrix using a shrinkage method and estimating the mismatch steering vector by solving a quadratic convex optimization problem. Unlike previous works, we relax the norm constraint of the desired signal steering vector in our approach, resulting in a more general model. Numerical examples demonstrate that our algorithm outperforms the existing adaptive beamforming techniques, not only in terms of output performance but also in terms of computational complexity.

2. THE SIGNAL MODEL

The output of a narrowband adaptive beamformer with $M$ sensors is given by

$$y(k) = w^H x(k)$$

(1)

where $k$ is the time index, $x(k) = [x_1(k), \ldots, x_M(k)]^T \in C^M$ is the array observation vector, $w = [w_1, \ldots, w_M]^T \in C^M$ is the complex vector of beamforming weights, and $(\cdot)^H$ denote the transpose and Hermitian transpose, respectively. The array observation vector $x(k)$ has the form

$$x(k) = x_s(k) + x_i(k) + x_n(k),$$

(2)

where $x_s(k) = a s(k)$, $x_i(k)$ and $x_n(k)$ are the statistically independent components of the desired signal, interference, and noise, respectively. In the desired signal term, $a \in C^M$ is the corresponding steering vector of desired signal $s(k)$.

The beamformer weight vector $w$ can be obtained by maximizing the signal-to-interference-plus-noise ratio (SINR)

$$\text{SINR} = \frac{\sigma_w^2 |w^H|^2}{w^H R_{i+n} w},$$

(3)

where $R_{i+n} = E\{[x_i(k) + x_n(k)](x_i(k) + x_n(k))^H\} \in C^{M \times M}$ is the interference-plus-noise covariance matrix, and

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\sigma^2 = E\{|s(k)|^2\} \text{ is the signal power. The optimization problem of maximizing array output SINR (3) can be equivalently written as the following minimum variance distortionless response (MVDR) problem}

$$\min_w w^H R_{i+n} w \text{ subject to } w^H a = 1,$$

and the solution is the well-known Capon beamformer

$$w_{opt} = \frac{R_{i+n}^{-1} a}{a^H R_{i+n}^{-1} a}. \quad (5)$$

Because the exact interference-plus-noise covariance matrix \(R_{i+n}\) is usually unavailable even in signal-free applications, the sample covariance matrix with \(K\) snapshots

$$\hat{R} = \frac{1}{K} \sum_{k=1}^K x(k)x^H(k) \quad (6)$$
is often adopted instead of \(R_{i+n}\). However, for small sample size, there is a large gap between \(\hat{R}\) and the theoretical covariance matrix \(R = \sigma^2 a a^H + R_{i+n}\). This gap will degrade the array performance. In addition, when \(R\) contains the desired signal, the adaptive beamformer needs many more snapshots to maintain its performance. Recently, a shrinkage method has been introduced to estimate the theoretical covariance matrix in the sense of minimizing mean square error (MMSE) [1], which is robust against small sample size but is helpless in the case of large steering vector mismatch.

Furthermore, in the presence of any steering vector mismatch, the Capon beamformer (5) is also no longer optimal, which might lead to a severe performance degradation. Although the adaptive beamforming approach based on the worst-case performance optimization [3] can be used to improve the robustness against the steering vector mismatch, the actual norm upper bound of the mismatch vector is \textit{a priori} unknown. In [5], an adaptive beamforming algorithm demonstrates its robustness against steering vector mismatch by iterative estimating the mismatch steering vector. However, this adaptive beamforming algorithm is unreliable when the sample size is small and/or the norm constraint of steering vector cannot be satisfied. Although the adaptive beamforming algorithm [7] combines the advantages of both [1] and [5], the norm constraint of the steering vector and the high computational complexity due to its iterative nature limit its applications.

### 3. THE PROPOSED ALGORITHM

In this section, a new adaptive beamforming algorithm is proposed, which is jointly robust against covariance matrix uncertainty and steering vector mismatch. The basic idea is to estimate the theoretical covariance matrix and the actual steering vector of the desired signal in sequence, not just one of them. To achieve this, the covariance matrix is first estimated based on a shrinkage method, which is subsequently used to estimate the mismatch vector between the actual steering vector and the presumed vector by solving a quadratic convex optimization problem.

To estimate the theoretical covariance matrix \(R\), a linear shrinkage estimator is given by [1]

$$\hat{R} = \beta \hat{R} + \alpha I, \quad (7)$$

where \(\alpha \geq 0\) and \(\beta \geq 0\) are the shrinkage parameters, and \(I\) is an identity matrix. By minimizing the mean square error (MSE) of the estimator \(\hat{R}\)

$$MSE(\hat{R}) = E\{|\hat{R} - R|^2\} = \alpha^2 M - 2\alpha(1 - \beta)tr(\hat{R}) + (1 - \beta)^2 \| \hat{R} \|^2 + \beta^2 E\{|\hat{R} - R|^2\}, \quad (8)$$

the optimal shrinkage parameters \(\alpha\) and \(\beta\) are finally given by

$$\hat{\alpha} = \min \left[ 1 - \frac{\| \hat{R} \|^2}{\| \hat{R} \|^2 - 1} \left( \frac{\| \hat{R} \|^2 + 3}{\| \hat{R} \|^2 - 1/\delta} \right)^{-1/2} \right], \quad (9)$$

$$\hat{\beta} = \frac{1 - \hat{\alpha}}{\hat{\nu}}, \quad (10)$$

where \(\hat{\nu} = \frac{tr(\hat{R})}{M}\), \(\hat{\rho} = \frac{1}{K} \sum_{k=1}^K \| x(k) \|^4 - \frac{1}{K} \| \hat{R} \|^2\).

Substituting \(\hat{\alpha}\) (9) and \(\hat{\beta}\) (10) into (7), we can get the shrinkage estimate of theoretical covariance matrix \(R\),

$$\hat{R} = \hat{\beta} \hat{R} + \hat{\alpha} I, \quad (11)$$

which belongs to the class of diagonal loading approaches with diagonal loading factor \(\hat{\alpha}/\hat{\beta}\). The shrinkage estimate \(\hat{R}\) will be used in the following estimation of the steering vector of desired signal.

In previous studies of robust adaptive beamforming, the norm constraint of the steering vector was usually assumed to be

$$\| a \|^2 = M. \quad (12)$$

Although it relaxes the requirement that the received signals from different sensors should maintain the same amplitude, this norm constraint is comparatively bold, because in practical applications (e.g., in wireless communications), this norm constraint usually cannot be satisfied mainly due to the effects of local scattering. Hence, this norm constraint will not be adopted, which means the steering vector can be generally modeled as

$$a = \left[ g_1 e^{j\varphi_1}, g_2 e^{j(\varphi_1 + \varphi)}, \ldots, g_M e^{j((M-1)\varphi + \varphi_M)} \right]^T \quad (13)$$

with individual amplitude \(g_m\) and phase disturbance \(\varphi_m\).

By substituting (5) back into the objective function of (4), the beamformer output power can be obtained as follows

$$P(e) = \frac{1}{(\bar{a} + e)^H \hat{R}^{-1} (\bar{a} + e)}, \quad (14)$$
where the mismatch vector $\mathbf{e}$ between the actual steering vector $\mathbf{a}$ and the presumed vector $\hat{\mathbf{a}}$, can be estimated by minimizing the denominator of (14). The mismatch vector $\mathbf{e}$ can be decomposed into two components. One denoted by $\mathbf{e}_\parallel$ is orthogonal to $\hat{\mathbf{a}}$, and the other denoted by $\mathbf{e}_\perp$ is parallel to $\hat{\mathbf{a}}$. That is to say, $\mathbf{e}_\perp$ is just a scale copy of $\hat{\mathbf{a}}$, which is immaterial because any scale of steering vector does not impact the output SINR. Therefore, similar to [5], the optimization problem of searching for $\mathbf{e}_\perp$ can be formulated as 

$$ \begin{align*} 
\min_{\mathbf{e}_\perp} & \quad (\hat{\mathbf{a}} + \mathbf{e}_\perp)^H \mathbf{R}^{-1} (\hat{\mathbf{a}} + \mathbf{e}_\perp) \\
\text{subject to} & \quad \hat{\mathbf{a}}^H \mathbf{e}_\perp = 0, (\hat{\mathbf{a}} + \mathbf{e}_\perp)^H \mathbf{V}_n = 0 \ \ (\hat{\mathbf{a}} + \mathbf{e}_\perp)^H \bar{\mathbf{C}} (\hat{\mathbf{a}} + \mathbf{e}_\perp) \leq \hat{\mathbf{a}}^H \bar{\mathbf{C}} \hat{\mathbf{a}} 
\end{align*} $$

(15)

where the first equality constraint is used to maintain the orthogonality between $\mathbf{e}_\perp$ and $\hat{\mathbf{a}}$. In the second equality constraint, $\mathbf{V}_n$ is an orthogonal matrix to the actual steering vector $\mathbf{a}$, and it is made of the noise eigenvectors of $\mathbf{R}$ corresponding to those minimum eigenvalues. Furthermore, the inequality constraint can be used to control the output power from the sidelobe regions with $\bar{\mathbf{C}} = \frac{1}{\Theta} \int_0^{\Theta} \mathbf{c}(\theta) \mathbf{c}^H(\theta) d\theta$, where $\mathbf{c}(\theta)$ is the steering vector associated with a potential direction $\theta$ and $\Theta$ combines a continuum of all out-of-sector directions.

The optimization problem (15) is a quadratic programming problem that can be easily solved using standard and highly efficient convex optimization software [8]. When $\mathbf{e}_\perp$ is solved, the presumed steering vector is corrected by 

$$ \hat{\mathbf{a}} = \mathbf{a} + \mathbf{e}_\perp. $$

(16)

Here, the normalization operation $\| \hat{\mathbf{a}} \| / \sqrt{\mathbf{M}} = 1$ adopted in [5] is unnecessary, because we do not know $a$ priori the norm of the steering vector from the general model (13), and the normalization operation of steering vector does not affect the beamformer output SINR.

It needs to be pointed out although the probability is very small, there is also a possibility of no feasible solution in (15) at low SNR, because the orthogonality between signal-plus-interference subspace and noise subspace breaks down due to subspace swaps. As we know, there is not much difference among existing adaptive beamformers at low SNR, except the eigen-decomposition based beamformer. Hence, a simple method is to let $\mathbf{e}_\perp = 0$ when the optimization problem (15) is infeasible, and then the proposed beamformer will be degraded to a shrinkage based beamformer [1].

In summary, the proposed beamforming algorithm performs the following steps:

**step 1**: Obtain $\hat{\mathbf{R}}$ (11) by calculating the shrinkage parameters $\hat{\alpha}_o$ (9) and $\hat{\beta}_o$ (10).

**step 2**: Solve the optimization problem (15) when it is feasible, otherwise let $\mathbf{e}_\perp = 0$.

**step 3**: Obtain the corrected steering vector $\hat{\mathbf{a}}$ (16).

**step 4**: Calculate the adaptive beamformer weights as 

$$ \mathbf{w} = \frac{\hat{\mathbf{R}}^{-1} \hat{\mathbf{a}}}{\hat{\mathbf{a}}^H \hat{\mathbf{R}}^{-1} \hat{\mathbf{a}}}. $$

(17)

Unlike [5], the iteration process of searching for the mismatch vector is unnecessary in our approach, which mainly benefits from the former shrinkage estimation of the covariance matrix. Therefore, the computational complexity of our method is much lower than those in [5, 7].

### 4. SIMULATION

In our simulations, a uniform linear array with $M = 10$ omnidirectional sensors spaced a half wavelength apart is considered. The additive noise is modeled as a complex Gaussian zero-mean spatially and temporally white process that has identical variances in each sensor. Two interfering sources are assumed to have DOAs $-50^\circ$ and $-20^\circ$, respectively. The interference-to-noise ratio (INR) in each sensor is equal to 30 dB. The desired signal is assumed to be a plane-wave from the presumed direction $\theta_o = 5^\circ$. The sample covariance matrix is computed based on $K = 100$ data snapshots. For each scenario, 200 Monte-Carlo runs are performed.

The proposed beamforming algorithm was compared to the sample matrix inversion (SMI) beamformer, the diagonal loading SMI (DLSMI) beamformer, the worst-case beamformer [3], the shrinkage based beamformer [1], the SQP based beamformer [5], and the shrinkage-plus-SQP based beamformer [4]. In the proposed algorithm, $\mathbf{V}_n$ is chosen to be the eigenvector of $\mathbf{R}$ corresponding to the minimal eigenvalue $\lambda_{min}$. In [4, 5], the parameters are set as $\delta = 0.1$, $K = 6$, and $\Theta = [\theta_p-5^\circ, \theta_p+5^\circ]$, respectively. The diagonal loading factor $\xi = 10\sigma_n^2$ is fixed in the DLSMI beamformer and the SQP based beamformer [5]. In the worst-case beamformer [3], the norm upper-bound $\epsilon = 0.3M$ is adopted as recommended. CVX software [8] was used to solve these convex optimization problems.

In the first example, a look direction mismatch of $3^\circ$ is assumed; i.e., the actual steering vector is calculated at $\theta = 8^\circ$. In addition, the amplitude of each sensor for both the desired signal steering vector and the interference steering vectors is normally distributed in $N(1, 0.1)$. The performance of all methods shown in Fig. 1 indicates that the proposed algorithm outperforms other algorithms tested.

In the second example, random sensor position errors, random look direction mismatch, and random amplitudes of steering vectors are considered together. Each sensor is assumed to be randomly displaced from its original location and the displacement is drawn uniformly from the set $[-0.05, 0.05]$ measured in wavelength. Also, the look direction mismatch is assumed to be random and uniformly distributed in $[-4^\circ, 4^\circ]$. In addition, the amplitude of each sensor for both the desired signal steering vector and the interference steering vectors is uniformly distributed in $[0.8, 1.2]$. The simulation results in Fig. 2 show that the proposed method outperforms other techniques.

Although there is no iteration requirement in our approach, the number of iterations in [5] and [7] are also pre-
presented in Fig. 3. From the output performance comparisons of Fig. 1 and Fig. 2, it is clear that the iteration process is unnecessary in our approach, which thus reduces its computational complexity.

5. CONCLUSION

This paper proposed a simple effective adaptive beamforming algorithm, which is robust against not only covariance matrix uncertainty but also steering vector mismatch. Based on a shrinkage estimation of the theoretical covariance matrix, the presumed steering vector can be corrected by searching for the mismatch steering vector via quadratic programming. The proposed algorithm does not need the norm constraint of the steering vector, which thus extends its applications. In addition, no iteration is required in the proposed method. The simulation results demonstrate the effectiveness of the proposed algorithm.

6. REFERENCES


