A NEAR-OPTIMAL LEAST SQUARES SOLUTION TO RECEIVED SIGNAL STRENGTH DIFFERENCE BASED GEOLOCATION

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ABSTRACT
A simple geometric interpretation for received signal strength (RSS) difference based geolocation can be illustrated by considering a plane containing a single pair of receivers and a transmitter. If the path loss follows a simple inverse power law, the RSS difference (in decibels) between the two receivers can be shown to define a circle on which the transmitter must lie. With additional receivers, the position of the transmitter can be solved by finding the common intersection of the circles corresponding to the different pairs of receivers. In practice, the solution of this problem is complicated by the errors contributed by environmental noise, measurement errors and the deviation of the actual path losses from the model. The optimal nonlinear least squares solution can be obtained by performing a search on a planar grid. However, the computational cost becomes an issue when the number of receivers is large. This paper presents an efficient solution whose computational cost is large. This paper presents an efficient solution whose performance approaches that of the optimal nonlinear least squares solution.

Index Terms—Received signal strength (RSS), empirical path loss models, emitter geolocation, linear least squares (LS) estimation, nonlinear least squares (NLS) estimation

1. EMPIRICAL PATH LOSS MODELS

Empirical path loss models are useful for estimating the received signal strength (RSS) as a function of parameters such as propagation distance, $d$, transmit antenna height, $h_t$, receive antenna height, $h_r$, and the carrier frequency of the transmitted signal, $f$, [1]-[3]. Although these models differ in details, the dependence of the RSS on the transmitter to receiver separation is generally expressed as a power law of the form:

$$P_r = \frac{A}{d^\gamma} P_t$$  \hspace{1cm} (1)

where $P_t$ is the transmitted signal power, $A = A(h_t, h_r, f)$ is a positive constant that depends on $h_r, h_t, f$ and $\gamma \geq 2$ is a positive constant, called the path loss exponent. Here $d > 0$ must be sufficiently large to avoid near-field effects. On a logarithmic scale, (1) can be reformulated as:

$$\Omega = 10 \log_{10}(P_r) = 10 \log_{10}(AP_t) - 10 \gamma \log_{10}(d) = C - 10 \gamma \log_{10}(d)$$ \hspace{1cm} (2)

where $C = C(h_t, h_r, f, P_t) = 10 \log_{10}(AP_t)$. If $h_t$, $h_r$, $P_t$ and $f$ are fixed, $C$ will essentially be a constant for all receivers [3]. This model is widely used in RSS based geolocation [2]-[4].

2. A GEOMETRIC INTERPRETATION OF RSS DIFFERENCE BASED GEOLOCATION

The possibility of geolocating a radio frequency emitter using RSS measurements obtained from simple sensors has motivated much recent research [3]-[5]. For uncooperative emitters, a priori knowledge of the transmitted signal power is not available and it is necessary to obtain solutions in terms of the relative RSSs observed at different locations. This approach has a simple geometric interpretation.

Consider an RF emitter at $T = (x, y)$ and two receivers at $R_1 = (x_1, y_1)$ and $R_2 = (x_2, y_2)$. Denote the distances from $R_1$ and $R_2$ to $T$ by $d_1$ and $d_2$, respectively, and the distance from $R_1$ to $R_2$ by $d_{12}$; i.e.,

$$d_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$ \hspace{1cm} (3)

Let the RSS (in dBm) at $R_1$ and $R_2$ be denoted by $\Omega_1$ and $\Omega_2$ respectively and define $\Omega_{12} = \Omega_1 - \Omega_2$. Then, according to (2),

$$\Omega_{12} = 10 \gamma \log_{10}(d_2/d_1)$$  \hspace{1cm} (4)

where the constant but unknown parameter $C$ cancels out. Let

$$\alpha_{12} = 10 \frac{\Omega_{12}}{\gamma}$$  \hspace{1cm} (5)

Equation (4) can then be rewritten as

$$(x-x_2)^2 + (y-y_2)^2 = \alpha_{12} [(x-x_1)^2 + (y-y_1)^2]$$  \hspace{1cm} (6)

If $\alpha_{12} \neq 1$, (6) can be rewritten as:

$$\left[\frac{x - x_2 - \alpha_{12}x_1}{1 - \alpha_{12}}\right]^2 + \left[\frac{y - y_2 - \alpha_{12}y_1}{1 - \alpha_{12}}\right]^2 = \frac{\alpha_{12}d_{12}^2}{(1 - \alpha_{12})^2}$$  \hspace{1cm} (7)
which represents a circle with radius \( \sqrt{d_{12}} \) and center

\[
C_{12} = \left[ \frac{x_2 - \alpha_{12} x_1}{1 - \alpha_{12}}, \frac{y_2 - \alpha_{12} y_1}{1 - \alpha_{12}} \right]
\]  

(8)

If \( \alpha_{12} = 1 \), (6) degenerates to the linear equation:

\[
(x_1 - x_2)x + (y_1 - y_2)y = \frac{x_1^2 + y_1^2 - x_2^2 - y_2^2}{2}
\]  

(9)

which represents the perpendicular bisector of the line segment passing through \( R_1 \) and \( R_2 \). In mathematics, straight lines and circles are considered as equivalent geometric entities under the stereographic projection. Thus the straight line (9) can also be treated as a circle.

It can be verified that each value of \( \alpha_{12} \) corresponds to a different non-intersecting circle and any point on the plane lies on a unique circle of the form (7) or (9). The one-to-one correspondence between \( \Omega_{12} \) and the associated circle is depicted in Fig. 1, where, as \( |\Omega_{12}| \to 0 \), the associated circle becomes larger and converges to the perpendicular bisector of the line segment \( R_1 R_2 \).

![Fig. 1. One-to-one correspondence between RSS differences \( \Omega_{12} \) and the associated circles. The receivers are located at \( R_1 \) and \( R_2 \).](image)

Since each pair of receivers generates a circle on which the emitter lies, the problem of localizing the RF emitter is reduced to that of finding the common intersection of multiple circles in some optimal manner. The circles associated with pairs of receivers are analogous to lines of bearing in angle of arrival (AOA) based geolocation. As an illustration, the circles associated with the 10 possible pairs of receivers out of a total of 5 receivers are plotted in Fig. 2, where the circles intersect at a single point at which the transmitter is located.

![Fig. 2. Transmitter at the intersection of circles associated with 10 pairs of receivers, assuming (2) is followed perfectly.](image)

**3. THE NONLINEAR LEAST SQUARES (NLS) SOLUTION**

In practice, the measured RSS will deviate from that predicted by (2) due to measurement errors and multipath propagation and shadowing effects. Consequently, the model (2) should be modified as follows:

\[
\bar{\Omega} = \Omega + Z = C - 10\gamma \log_{10}(d) + Z
\]  

(10)

where \( \bar{\Omega} \) denotes the measured RSS and \( Z \) is a zero-mean random variable with variance \( \sigma^2 \). Now consider \( n \) receivers located at \( R_k = (x_k, y_k) \), \( 1 \leq k \leq n \). The measured RSS at \( R_k \), denoted by \( \bar{\Omega}_k \) here, can be written as:

\[
\bar{\Omega}_k = C - 10\gamma \log_{10}(d_k) + Z_k, \quad 1 \leq k \leq n
\]  

(11)

where \( d_k \) denotes the Euclidean distance from the transmitter to the receiver at \( R_k \), \( 1 \leq k \leq n \), and \( Z_k \), \( 1 \leq k \leq n \), are i.i.d. zero-mean random variables with variance \( \sigma^2 \). Let \( \bar{\Omega}_{kl} = \bar{\Omega}_k - \bar{\Omega}_l \). Then

\[
\bar{\Omega}_{kl} = 10\gamma \log_{10}(d_k/d_l) + Z_{kl}
\]

\[
= 5\gamma \log_{10} \left[ \frac{(x - x_l)^2 + (y - y_l)^2}{(x - x_k)^2 + (y - y_k)^2} \right] + Z_{kl}
\]  

(12)

where \( (x, y) \) is the location of the transmitter and \( Z_{kl} \), \( 1 \leq k < l \leq n \), are zero-mean random variables with variance \( 2\sigma^2 \). Let

\[
\Delta_n = \{(k, l) \mid 1 \leq k < l \leq n\}
\]  

(13)

The lexical order \(<\) on the set \( \Delta_n \) is defined by the rule that \( (p, q) < (s, t) \) if and only if either \( p < s \) or if \( p = s \), then \( q < t \). Let \( \Delta \subset \Delta_n \) be a subset of the set of all receiver pairs. The nonlinear least squares (NLS) estimate of \( (x, y) \) based on the receiver pairs specified by \( \Delta \), denoted by \( (\hat{x}_\Delta, \hat{y}_\Delta) \)
here, can be obtained by minimizing the objective function $Q_\Delta(x, y)$ defined by:
\[
Q_\Delta(x, y) = \sum_{(k,l) \in \Delta} \left( \bar{\Omega}_{kl} - 5\gamma \log_{10} \left( \frac{(x-x_k)^2 + (y-y_k)^2}{(x-x_k)^2 + (y-y_k)^2} \right) \right)^2
\]
(14)
or alternatively,
\[
(\hat{x}_\Delta^1, \hat{y}_\Delta^1) = \text{argmin } Q_\Delta(x, y)
\]
(15)

4. A LEAST SQUARES (LS) SOLUTION

The objective function $Q_\Delta$ is a nonlinear multi-modal function and the only reliable method to compute $(\hat{x}_\Delta^1, \hat{y}_\Delta^1)$ is to form a large refined grid and minimize the function $Q_\Delta$ on this grid, a procedure that is computationally intensive. A more efficient approach for estimating the transmitter location $(x, y)$ has been obtained in [5] by linearizing the nonlinear system (12). If the noise term $Z_{kl}$ in (12) is ignored, we obtain
\[
\hat{\Omega}_{kl} = 5\gamma \log_{10} \left( \frac{(x-x_k)^2 + (y-y_k)^2}{(x-x_k)^2 + (y-y_k)^2} \right)
\]
(16)
Setting $\hat{\alpha}_{kl} = 10^{\hat{\Omega}_{kl}/5}$, equation (16) can be rewritten as:
\[
(x-x_k)^2 + (y-y_k)^2 = \hat{\alpha}_{kl} \left[ (x-x_k)^2 + (y-y_k)^2 \right]
\]
(17)
or
\[
(1 - \hat{\alpha}_{kl})(x^2 + y^2) + 2(\hat{\alpha}_{kl}x_k - x_k)x + 2(\hat{\alpha}_{kl}y_k - y_k)y = w_{kl}, \quad 1 \leq k < l \leq n
\]
(18)
where $w_{kl} = \hat{\alpha}_{kl}l_{kl}^2 - r_{km}^2$, $r_{m} = \sqrt{2x_m^2 + y_m^2}$, $1 \leq m \leq n$. To solve for $x$ and $y$, an interesting idea introduced in [5] is to linearize the system (18) by introducing a new parameter $c = x^2 + y^2$ and treat it as if it were independent of $x$ and $y$. With the introduction of the parameter $c$, the quadratic system (18) is transformed into the following linear system in $c, x, y$:
\[
(1 - \hat{\alpha}_{kl})c + u_{kl}x + v_{kl}y = w_{kl}, \quad 1 \leq k < l \leq n
\]
(19)
where $u_{kl} = 2(\hat{\alpha}_{kl}x_k - x_k)$ and $v_{kl} = 2(\hat{\alpha}_{kl}y_k - y_k)$, $1 \leq k < l \leq n$. Now assume $\Delta \subset \Delta_n$ is a subset of $\Delta_n$ with cardinality $m = |\Delta|$. Define
\[
\nu_{kl} = (1 - \hat{\alpha}_{kl}, u_{kl}, v_{kl})
\]
(20)
and let $A_\Delta$ be the $m \times 3$ matrix consisting of the $m$ row vectors $\nu_{kl}$ associated with the receiver pairs specified by $\Delta$ and arranged in increasing lexical order and let $b_\Delta$ be the $m \times 1$ matrix consisting of the $m$ terms $w_{kl}$ specified by $\Delta$ and arranged in the same lexical order. The LS solution to the subsystem of (19) obtained by confining $(k, \ell) \in \Delta$, denoted by $(\hat{c}, \hat{x}, \hat{y})$ here, is computed by:
\[
(\hat{c}, \hat{x}, \hat{y}) = \left( (A_\Delta)^{t} A_\Delta \right)^{-1} (A_\Delta)^{t} b_\Delta
\]
(21)
where $(A_\Delta)^{t}$ denotes the matrix transpose of $A_\Delta$. An estimate of the transmitter location $(x, y)$, denoted by $(\hat{x}_\Delta^2, \hat{y}_\Delta^2)$ here, is then obtained by setting:
\[
(\hat{x}_\Delta^2, \hat{y}_\Delta^2) = (\hat{x}, \hat{y})
\]
(22)

5. A CLOSED FORM LS SOLUTION

A simpler closed form LS solution can be formulated by deriving a linear system in $x, y$ from the quadratic system (18) without introducing the dummy variable $c = x^2 + y^2$. Equation (18) can be rewritten as:
\[
x^2 + y^2 + a_{kl}x + b_{kl}y = c_{kl}, \quad 1 \leq k < l \leq n
\]
(23)
where
\[
a_{kl} = \frac{u_{kl}}{1 - \alpha_{kl}}, \quad b_{kl} = \frac{v_{kl}}{1 - \alpha_{kl}}, \quad c_{kl} = \frac{w_{kl}}{1 - \alpha_{kl}}
\]
(24)
Consider $((p, q), (s, t)) \in \Gamma$ where the set $\Gamma_n$ is defined by:
\[
\Gamma_n = \{ ((p, q), (s, t)) | (p, q) \times (s, t), (p, q), (s, t) \in \Delta_n \}
\]
(25)
Subtracting the following two equations
\[
\left\{ \begin{array}{ll}
x^2 + y^2 + a_{pq}x + b_{pq}y &= c_{pq} \\
x^2 + y^2 + a_{st}x + b_{st}y &= c_{st}
\end{array} \right.
\]
(26)
yields a linear equation in $x$ and $y$:
\[
\mu(p, q, s, t)x + \nu(p, q, s, t)y = \tau(p, q, s, t)
\]
(27)
where
\[
\left\{ \begin{array}{ll}
\mu(p, q, s, t) &= a_{pq} - a_{st} \\
\nu(p, q, s, t) &= b_{pq} - b_{st} \\
\tau(p, q, s, t) &= c_{pq} - c_{st}
\end{array} \right.
\]
(28)
Let $\Gamma \subset \Gamma_n$ be a subset of $\Gamma_n$ and let the LS solution to the subsystem of (27) specified by $\Gamma$ be denoted by $(\hat{x}_\Gamma, \hat{y}_\Gamma)$. Then it can be shown that
\[
\left\{ \begin{array}{ll}
\hat{x}_\Gamma &= \frac{T_{\mu\mu}T_{\nu\nu} - T_{\mu\nu}T_{\mu\nu}}{T_{\mu\mu}T_{\nu\nu} - T_{\mu\nu}T_{\mu\nu}} \\
\hat{y}_\Gamma &= \frac{T_{\mu\mu}T_{\nu\nu} - T_{\mu\nu}T_{\mu\nu}}{T_{\mu\mu}T_{\nu\nu} - T_{\mu\nu}T_{\mu\nu}}
\end{array} \right.
\]
(29)
where
\[
\left\{ \begin{array}{ll}
T_{\mu\mu} &= \sum_{((p, q), (s, t)) \in \Gamma} \mu(p, q, s, t)^2 \\
T_{\nu\nu} &= \sum_{((p, q), (s, t)) \in \Gamma} \nu(p, q, s, t)^2 \\
T_{\mu\nu} &= \sum_{((p, q), (s, t)) \in \Gamma} \mu(p, q, s, t)\nu(p, q, s, t) \\
T_{\mu\tau} &= \sum_{((p, q), (s, t)) \in \Gamma} \mu(p, q, s, t)\tau(p, q, s, t) \\
T_{\nu\tau} &= \sum_{((p, q), (s, t)) \in \Gamma} \nu(p, q, s, t)\tau(p, q, s, t)
\end{array} \right.
\]
(30)
Note that here we have implicitly assumed that, for each receiver pair \((k, l)\), \(1 \leq k < l \leq n\), \(1 - \bar{\alpha}_{kl} \neq 0\). To avoid potential numerical problems arising from division by \(1 - \bar{\alpha}_{kl}\) in (24), without loss of generality, the coefficients \(\mu, \nu\) and \(\tau\) of equation (27) should be modified as follows:

\[
\begin{align*}
\mu(p, q, s, t) &= (1 - \bar{\alpha}_{st})w_{pq} - (1 - \bar{\alpha}_{pq})w_{st} \\
\nu(p, q, s, t) &= (1 - \bar{\alpha}_{st})v_{pq} - (1 - \bar{\alpha}_{pq})v_{st} \\
\tau(p, q, s, t) &= (1 - \bar{\alpha}_{st})w_{pq} - (1 - \bar{\alpha}_{pq})w_{st}
\end{align*}
\]

Equation (27) has the geometric interpretation that it represents the straight line passing through the two intersections of the two circles defined by (26). Instead of computing the common intersection of the circles (23), which does not have a simple mathematical solution, an easier way is to compute the intersection of the straight lines (27). This idea forms the basis of the closed form LS solution (29).

6. PERFORMANCE COMPARISON

The performance of the estimators (15), (22) and (29) depends strongly on the sets \(\Delta\) and \(\Gamma\) selected. Since \(\Omega_{kl} = \Omega_{ul} - \Omega_{1k}\), \(1 < k < l \leq n\), the \(n - 1\) random variables, \(\bar{\Omega}_{kl}\), \(2 \leq k \leq n\), should contain most of the useful information for geolocation. Therefore, it suffices to focus on the estimators specified by the following small sets:

\[
\Delta^* = \{(k, l) \mid k = 1, 2 \leq l \leq n\}
\]

\[
\Gamma_1^* = \{((p, q), (s, t)) \mid p = 1, s = 1, q = 2, 3 \leq t \leq n\}
\]

\[
\Gamma_2^* = \{((p, q), (s, t)) \mid p = 1, s = 1, 2 \leq q < t \leq n\}
\]

Extensive simulations have shown that the LS solution (22) defined by \(\Delta^*\) and the closed form LS solution (29) defined by \(\Gamma_1^*\) produce identical results. However, the LS solution (22) defined by \(\Delta^*\) and the closed form LS solution (29) defined by \(\Gamma_1^*\) in general have different performance as the emitter-sensor geometry varies. As an illustration, in Fig. 3, the root mean square error of the estimated transmitter location is plotted as a function of \(\sigma\) for the scenario where a transmitter is located at \((0, 0)\) and 10 power sensors are located at \((-126, 62), (-339, 166), (390, 76), (-178, 333), (482, 122), (-234, 141), (230, -45), (-352, 279), (-104, 465)\) and \((366, -139)\). The path loss exponent is set at \(\gamma = 4\) and the additive noise terms \(Z_k\) of the RSS are assumed to have a zero-mean Gaussian distribution with equal standard deviation \(\sigma\) for each sensor. The search grid used in the NLS solution (15) is defined on a square centered at \((0, 0)\) and of length 1000 meters on each side with a 2 meter grid spacing. 1000 simulation runs were performed for each value of \(\sigma\). For this particular placement of emitter and sensors, for smaller values of \(\sigma\), the performance of the proposed closed-form LS solution (29) approaches that of the optimal NLS solution. However, for larger values of \(\sigma\), the RMSE curve for the closed-form LS solution (29) is actually below that for the optimal NLS solution (15).

The interpretation of these results involves some subtleties. In particular, there are practical implementation issues with the NLS and other geolocation algorithms that involve a search over a 2 dimensional grid. For these algorithms, arbitrary constraints are often imposed on the size of the grid to minimize computational cost. Since this constrains the allowed position estimates, the effect is to introduce a bias in the position estimates that depends on the relative emitter position within the grid, the grid dimensions and \(\sigma\). Further research is needed to better understand the relative merits and limitations of these algorithms.

Fig. 3. Simulation results for the closed-form LS solution (29) defined by \(\Gamma_1^*\), the LS solution (22) defined by \(\Delta^*\) and the optimal NLS solution (15) defined by \(\Delta_n^*\).

7. REFERENCES


