Gain and Phase Autocalibration for Uniform Rectangular Arrays

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Abstract—To maintain the performance of direction-of-arrival (DOA) estimation, an accurate model of the array response is required. In a time-varying sensor environment, this is only possible with autocalibration. For a uniform linear array, there exist algorithms for autocalibration which exploit the Toeplitz structure of the unperturbed spatial covariance matrix. In this paper, we develop an autocalibration method for 2-D DOA estimation with a uniform rectangular array, in which we exploit a Toeplitz-block Toeplitz structure. We present a simple algorithm for gain and phase estimation, discuss ambiguity problems and evaluate the performance using simulations.

Index Terms—Autocalibration, two-dimensional DOA estimation, uniform rectangular array

I. INTRODUCTION

Multiple studies have shown that high-resolution methods for direction-of-arrival (DOA) estimation are sensitive to model errors, e.g. [1] or [2]. A real sensor array usually implicates several systematic errors, which are mutual coupling, gain and phase errors or position errors. To ensure a certain estimation performance we require an accurately known model of the sensor array, which is often found using off-line calibration measurements with single targets of known location, see e.g. Chapter 3 in [3]. If calibration measurements are not available, e.g. due to cost reasons, and the array model is only known imperfectly, autocalibration becomes necessary. It is of particular interest for practical array processing when the sensor environment (temperature, humidity, etc.) is time-varying.

We remark, however, that autocalibration is only useful if the array processing performance error caused by model imperfections is of similar magnitude or larger than the error caused by a finite SNR and few snapshots [4].

For a uniform linear array (ULA), a group of algorithms performs autocalibration by exploiting the Toeplitz structure of an unperturbed spatial covariance matrix, e.g. [5] or [6]. This principle can also be applied in a simpler form, i.e. by only utilizing the information on the main diagonal for gain estimation, and on the first superdiagonal for phase estimation [7]. This simplification has been analyzed in [8].

The contribution of this paper is to develop autocalibration methods for 2-D DOA estimation with a uniform rectangular array (URA). We follow the idea in [5] for the ULA, and exploit in our case the Toeplitz-block Toeplitz structure of the unperturbed covariance matrix. We present a computationally simple algorithm for gain and phase estimation, discuss ambiguity problems and propose an adequate remedy. A further contribution is the extension of the Friedlander-Weiss method to the 2-D case. The performance of the presented algorithms is evaluated using simulations.

II. SIGNAL MODEL

If $K$ narrowband sources impinge on an array with $M$ sensor elements, the common baseband model for the array output vectors is given by

$$x(t) = \sum_{k=1}^{K} a(\psi_k)s_k(t) + n(t), \quad t = 1, \ldots, N$$

where $N$ is the number of snapshots available, $n(t)$ is independent random disturbance, $s_k(t)$ and $\psi_k$ are the source waveform and DOA parameter of source $k$, respectively. The steering vector $a(\psi_k)$ represents the nominal array response to a source at $\psi_k$, where $\psi = (\phi, \theta)$ denotes the pair of azimuthal and elevation angle, $\phi$ and $\theta$, respectively. The array geometry is shown in Figure 1.

![Array geometry of the URA.](image)

We consider a URA with $M_x$ and $M_y$ elements in $x$-direction and $y$-direction, respectively, resulting in $M = M_xM_y$ array elements. Following the notation in [9], the steering vector of a URA can be written as

$$a(\psi) = a_x(\psi) \otimes a_y(\psi)$$

where $\otimes$ denotes the Kronecker product. The elements of vectors $a_x(\psi)$ and $a_y(\psi)$ are given by

$$a_x(\psi) = \frac{1}{\sqrt{M_x}} e^{j\kappa d_x (m_x-1) u_x(\psi)},$$

$$a_y(\psi) = \frac{1}{\sqrt{M_y}} e^{j\kappa d_y (m_y-1) u_y(\psi)},$$

for $m_x = 1, \ldots, M_x$ and $m_y = 1, \ldots, M_y$, respectively. The wavenumber is $\kappa = 2\pi/\lambda$, where $\lambda$ is the wavelength. The terms $u_x(\psi) = \sin \theta \cos \phi$ and $u_y(\psi) = \sin \theta \sin \phi$ are referred to as directional cosines.
In the following, we assume the source and noise signals to be independent, the noise to be spatially white with zero mean and a common variance \( \sigma_t^2 \). Furthermore, the source waveforms are assumed to be mutually uncorrelated with zero mean and variances \( \nu_k^2 \). The nominal spatial covariance matrix then is
\[
R = E \{ x(t)x(t)^H \} = \sum_{k=1}^{K} \nu_k^2 a(\psi_k)a(\psi_k)^H + \sigma_t^2 I
\]  
where \( E \{ \cdot \} \) denotes the expectation operator. Let us have a closer look at the rank-one term
\[
C(\psi) = a(\psi)a(\psi)^H = [a_y(\psi) \otimes a_x(\psi)] [a_y(\psi) \otimes a_x(\psi)]^H
= a_y(\psi)a_y(\psi)^H \otimes a_x(\psi)a_x(\psi)^H
= C_y(\psi) \otimes C_x(\psi)
\]  
where matrices \( C_x(\psi) \) and \( C_y(\psi) \) have elements
\[
[C_x(\psi)]_{m,n} = 1/M_x \cdot e^{j\varphi_x(m-n)}u_x(n),
[C_y(\psi)]_{m,n} = 1/M_y \cdot e^{j\varphi_y(m-n)}u_y(n).
\]
We observe that the elements of \( C_x(\psi) \) and \( C_y(\psi) \) are only a function of \( m_x-n_x \) or \( m_y-n_y \), so that the respective matrices have a Toeplitz structure. Consequently, based on Eq. (4), \( C \) has a Toeplitz-block Toeplitz structure, with \( M_y \times M_y \) blocks of size \( M_x \times M_x \). Applying this to Eq. (3), we observe that \( R \) is a sum of \( K \) Toeplitz-block Toeplitz matrices \( C(\psi_k) \) and an identity matrix, and therewith also has a Toeplitz-block Toeplitz structure.

III. AUTOCALIBRATION ALGORITHM

Considering gain and phase errors, the perturbed array output model is given by
\[
\tilde{x}(t) = \sum_{k=1}^{K} \Gamma a(\psi_k) s_k(t) + n(t), \quad t = 1, \ldots, N
\]  
with \( \Gamma = diag\{g_1 e^{j\varphi_1}, \ldots, g_M e^{j\varphi_M}\} \), where \( g_k \) and \( \varphi_k \) are the sensor gain and phase parameters, respectively, and \( diag\{\cdot\} \) is the diagonal matrix operator. Based on Eq. (3), (4) and (5), the perturbed covariance matrix is
\[
\tilde{R} = \Gamma \left\{ \sum_{k=1}^{K} \nu_k^2 C(\psi_k) \right\} \Gamma^H + \sigma_t^2 I
\]  
If we assume a sufficient SNR (otherwise autocalibration is not useful) such that \( \sigma_t^2 \) can be neglected when compared to \( \nu_k^2 \), the elements of \( \tilde{R} \) are given by
\[
[\tilde{R}]_{m,n} \approx g_m g_n e^{j(\varphi_m - \varphi_n)} [\tilde{R}]_{m,n}
\]
So by exploiting the Toeplitz-block Toeplitz property of \( R \) we are able to estimate the unknown gain and phase parameters \( \{g_k, \varphi_k\} \). Since \( \tilde{R} \) is not available in practice, it can be estimated using the sample covariance \( \tilde{R} = \sum_{t=1}^{N} \tilde{x}(t)\tilde{x}(t)^H \).

We remark that without the knowledge of the source waveforms, it is only possible to determine the gain and phase parameters up to a complex scalar. Consequently, we set \( g_1 = 1 \) and \( \varphi_1 = 0 \), and search for the remaining (normalized) gain and phase parameters.

A. Gain estimation

The following is a straightforward extension of [7]. Since the elements of \( R \) are equal on the main diagonal, we can construct \( M-1 \) linear equations for \( m = 1, \ldots, M-1 \)
\[
2 \log g_{m+1} - 2 \log g_m = \hat{\nu}_m + \epsilon_m
\]  
where \( \hat{\nu}_m = \log |[\tilde{R}]_{m+1,m+1} - |[\tilde{R}]_{m,m}| \) is the measured log difference of absolute values and \( \epsilon_m \) is a fitting error. In matrix notation, we have \( T\theta = \bar{\nu} + \epsilon \) where \( T \) is an \((M-1) \times (M-1)\) integer-valued system matrix of full rank, which can be easily obtained from Eq. (7), the real-valued parameter vector is given by \( \theta = [\log g_2, \ldots, \log g_M]^T \) and the elements of \( \bar{\nu} \) are as defined above. The solution which minimizes \( \|\epsilon\| \) in the least-squares sense is given by
\[
\hat{\theta} = (T^T T)^{-1} T^T \bar{\nu}
\]

B. Phase estimation

Before describing the phase estimation principle, we comment on the rotational ambiguity of phase error parameters, which is due to the Vandermonde structure of vectors \( a_x(\psi) \) and \( a_y(\psi) \). Let us consider the special case \( \Gamma = \Gamma_y \otimes I_x \).
\[
\Gamma a(\psi) = \Gamma_y a_y(\psi) \otimes \Gamma_x a_x(\psi) = \Gamma_y B_y (B_y^{-1} a_y(\psi)) \otimes \Gamma_x B_x (B_x^{-1} a_x(\psi))
\]
where \( a(\psi) \) is as defined in Eq. (2) and
\[
B_x = diag\{1, e^{j\beta_x}, \ldots, e^{j(M_x - 1)\beta_x}\},
B_y = diag\{1, e^{j\beta_y}, \ldots, e^{j(M_y - 1)\beta_y}\}.
\]
Consequently, the phase parameters \( \{\varphi_m\} \) cannot be identified uniquely together with the DOA parameters \( \{\psi_k\} \), which is due to rotational factors \( \beta_x \) and \( \beta_y \). Additional constraints are required to obtain a unique solution.

For constructing equations exploiting the Toeplitz-block Toeplitz structure of \( R \), we follow the idea in [7] and [8], and select only elements from \( \tilde{R} \) where either \( m_x - n_x \) or \( m_y - n_y \) equal zero. Three groups of relationships have been identified, and are described below. In what follows, we use the indexing variable \( m = m_x + (m_y - 1)M_x \), and \( \delta_m^{(i)} \) and \( \xi_m^{(i)} \) for \( i = 1, 2 \) and 3 to denote the measured phase differences and fitting errors of group \( i \), respectively.

Group 1: Relations within blocks, on the first superdiagonal, for \( m_x = 1, \ldots, M_x - 2 \) and \( m_y = 1, \ldots, M_y \)
\[
\varphi_m - 2\varphi_{m+1} + \varphi_{m+2} = \hat{\delta}_m^{(1)} + \epsilon_m^{(1)}
\]  
with \( \hat{\delta}_m^{(1)} = \angle [\tilde{R}]_{m+1,m+2} - \angle [\tilde{R}]_{m,m+1} \).

Group 2: Relations from block-to-block, on the first superdiagonal, for \( m_x = 1 \) and \( m_y = 1, \ldots, M_y - 1 \)
\[
\varphi_m - \varphi_{m+1} - \varphi_{m+M_x} + \varphi_{m+M_x+1} = \hat{\delta}_m^{(2)} + \epsilon_m^{(2)}
\]  
with \( \hat{\delta}_m^{(2)} = \angle [\tilde{R}]_{m,m+M_x,m+M_x+1} - \angle [\tilde{R}]_{m,m+1} \).

Group 3: Relations from block-to-block, on superdiagonal \( M_x \), for \( m_x = 1 \) and \( m_y = 1, \ldots, M_y - 2 \)
\[
\varphi_m - 2\varphi_{m+M_x} + \varphi_{m+2M_x} = \hat{\delta}_m^{(3)} + \epsilon_m^{(3)}
\]  
Where \( \hat{\delta}_m^{(3)} = \angle [\tilde{R}]_{m,m+M_x,M_x+1} - \angle [\tilde{R}]_{m,m+1} \).
with $\vartheta^{(3)}_m = \angle \mathbf{R}_{m+Mx,m+2Mx} - \angle \mathbf{R}_{m,m+Mx}$.

To illustrate the described groups of relations, the involved matrix elements are shown in the plot in Figure 2 for an example with $M_x = M_y = 4$. We have $(M_x - 2)M_y$, $(M_y - 1)$ and $(M_y - 2)$ relations, respectively, resulting in $M - 3$ linearly independent equations. We remark, that no matter how many relations are considered, the system matrix will at most have rank $M - 3$. As indicated above, this is due to the fact that $\beta_1$, $\beta_2$ and $\varphi_1$ cannot be identified without ambiguity. We remark also that generally more relations can be considered; this is not discussed here due to lack of space.

Analogously, writing Eq. (8), (9) and (10) in matrix notation, we obtain $Q\varphi = \hat{\vartheta} + \xi$ where $Q$ is an $(M - 3) \times (M - 1)$ integer-valued system matrix of full rank, which can be easily obtained from the equations above, the real-valued parameter vector is given by $\varphi = [\varphi_2, \ldots, \varphi_M]^T$ and the elements of $\hat{\vartheta}$ are as defined above. A 2-D equivalent of the zero mean phase error assumption as in [6], is to enforce $\beta_x = \beta_y = 0$. We choose an alternative approach and search for the smallest $\varphi$ which satisfies $Q\varphi = \hat{\vartheta}$, i.e.

$$\min_{\varphi} \varphi^T \varphi \text{ s.t. } Q\varphi - \hat{\vartheta} = 0$$

Using Lagrange multipliers, the solution can be shown to be

$$\hat{\varphi} = Q^T (QQ^T)^{-1} \hat{\vartheta}$$

C. Other algorithms

To compare our proposed algorithm with other approaches, we use an intuitive approach for $K = 1$, which is based on the conventional beamformer (BF) and matching the steering vector, and a 2-D modification of the Friedlander-Weiss method [10], which is applicable for $K > 1$.

The BF matching is described as follows: estimate the source waveform $\hat{s}(t) = a(\psi)^H \hat{x}(t)$, where $\psi$ is the maximum of the BF spectrum $a(\psi)^H \hat{R} a(\psi)$. Then estimate an unstructured array response vector using $\hat{\alpha} = \frac{1}{A} \sum_{t=1}^N \hat{s}(t)^* x(t)$ with $A = \sum_{t=1}^N |\hat{s}(t)|^2$, and determine the desired correction term by $[\hat{\Gamma}]_{mm} = [\hat{\alpha}]_m / [a(\psi)]_m$. We remark that this approach assumes the gain and phase errors to be sufficiently small such that the BF spectrum is not perturbed significantly.

The Friedlander-Weiss method is based on the relaxation principle of a joint estimation problem, i.e., iteratively estimating one group of parameters while keeping the other constant. The principle of [10] is summarized as follows: after initializing with $\gamma = 1$, a two-step procedure is used to iteratively minimize a MUSIC-type cost function

$$J(\psi, \gamma) = \sum_{k=1}^K a(\psi_k)^H \hat{E}_n \hat{E}_n^H \Gamma a(\psi_k)$$

(11)

with parameters $\psi = [\psi_1, \ldots, \psi_K]$ and $\Gamma = \text{diag}\{\gamma\}$, where $\hat{E}_n$ is the noise subspace of $\hat{R}$.

Step 1: Obtain $\psi$ by the $K$ largest peaks of the MUSIC spectrum $[a(\psi)]^H \Gamma^H \hat{E}_n \hat{E}_n^H \Gamma a(\psi)^{-1}$.

Step 2: Hold $\psi$ fixed and minimize $J(\psi, \gamma)$ w.r.t. $\gamma$. A reformulation of Eq. (11)

$$\gamma^H \sum_{k=1}^K \text{diag}\{a(\psi_k)\}^H \hat{E}_n \hat{E}_n^H \text{diag}\{a(\psi_k)\}$$

and a linear constraint $\delta^T \gamma = 1$ with $\delta = [1, 0, \ldots, 0]^T$ is employed. The solution then simply is $\gamma = M^{-1} \delta / (\delta^T M^{-1} \delta)$.

The two steps are repeated until a stopping criterion is reached, e.g. iteration index increases a bound or $J_{i-1} - J_i$ in (11) is smaller than a predefined threshold. We remark that this method is generally applicable for arbitrary array geometries. Although, the original paper [10] imposes a non-restrictive assumption of a non-linear array geometry, it can also be used with linear array geometries, provided a strategy to constrain the ambiguity is employed.

IV. SIMULATION

For the simulations, we model the array elements’ gain and phase errors for $m = 2, \ldots, M$ as

$$g_m \sim \mathcal{N}_\log(0 \, \text{dB}, 1 \, \text{dB}^2)$$

and $\varphi_m \sim \mathcal{U}(-\varphi_{\max}, \varphi_{\max})$ where we use $\mathcal{N}_\log(\mu, \sigma^2)$ to denote a log-normal distribution with mean $\mu$ and variance $\sigma^2$ and $\mathcal{U}(a, b)$ to denote a uniform distribution between $a$ and $b$. $\varphi_{\max}$ is the maximum phase error. If not otherwise stated, we consider a URA with $M_x = M_y = 4$ elements, spaced by $\lambda/2$.

A. Single source

To investigate the array error estimation accuracy, we compare the proposed algorithm with the BF matching approach, as described above. 1000 Monte-Carlo runs of a single source from $\theta = 30^\circ$ and $\phi = 50^\circ$ with $N = 64$ snapshots are simulated. Figure 3 shows the RMSE of $\|\hat{\Gamma} - \Gamma\|$ versus SNR (left) and versus $\varphi_{\max}$ (right). It can be observed that both the proposed algorithm and the BF matching approach are able to approximate the array gain and phase errors. At high SNR and for small phase errors $\varphi_{\max}$, the proposed algorithm yields better results than the BF matching approach.
B. Multiple sources

To investigate the DOA estimation accuracy, we compare the proposed algorithm with the Friedlander-Weiss method. We simulate two uncorrelated sources at $\phi = [60^\circ, 80^\circ]$ and $\theta = [45^\circ, 40^\circ]$ at SNR = 30 dB with $N = 64$ snapshots, the maximum phase error is $\varphi_{\text{max}} = 20^\circ$. Figure 4 shows a scatter plot of 250 independent DOA estimates using no correction, the Friedlander-Weiss method, the proposed Toeplitz-block Toeplitz autocalibration and the ideal correction. For all methods, the 2-D DOA estimation has been done using Unitary ESPRIT [11]. It can be observed that the proposed algorithm improves the estimation when compared to no correction.

In another simulation setup with 1000 runs, we use the same source locations as above, but an array with $M_x = M_y = 8$ sensors and $N = 256$ snapshots. Figure 5 shows the RMSE of RMSE $\varphi_{\text{max}}$, $\varphi_{\text{max}}$ [deg] at $\varphi_{\text{max}} = 20^\circ$, $\varphi_{\text{max}} = 30^\circ$, $\varphi_{\text{max}} = 40^\circ$, $\varphi_{\text{max}} = 50^\circ$, $\varphi_{\text{max}} = 60^\circ$. It can be observed that the proposed algorithm exploits a Toeplitz-block Toeplitz structure. We have proposed a computationally simple algorithm for gain and phase estimation. Furthermore, we have suggested a 2-D extension of the Friedlander-Weiss method. The latter method is iterative and based on a grid search, and therefore less feasible in practice. A performance analysis has shown that the DOA estimation accuracy is limited by a remaining phase error, which cannot be identified with a URA.

V. Conclusion

In this paper, we have presented an autocalibration method for 2-D DOA estimation with a URA. The idea is to exploit a Toeplitz-block Toeplitz structure. We have proposed a computationally simple algorithm for gain and phase estimation. Furthermore, we have suggested a 2-D extension of the Friedlander-Weiss method. The latter method is iterative and based on a grid search, and therefore less feasible in practice. A performance analysis has shown that the DOA estimation accuracy is limited by a remaining phase error, which cannot be identified with a URA.

References