A NEW APPROACH FOR THE ESTIMATION OF THE POROSITY IN NMR
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ABSTRACT
In this paper we describe a new approach for the estimation of the porosity and its uncertainty from Nuclear Magnetic Resonance relaxation measurements in porous media. The new approach is based on the Fourier transform of the measured data in √T domain. This approach was found to work reasonably well and had a smaller bias and variance in comparison to traditional methods of computing the porosity.

1. INTRODUCTION
Nuclear Magnetic Resonance (NMR) relaxation measurements are often used to probe the pore size distribution and fluid properties of hydrocarbons in porous media [1]. The measured data M(t) are modeled by means of the Laplace transform of the probability density function f_T(T)

$$M(t) = \int_0^\infty e^{-t/T} f_T(T) dT, \quad t = 1, 2, \ldots, N$$ (1)

where T is referred to as the relaxation time and assumed to be a continuum and t_E is referred to as the echo spacing. The non-negative amplitude f_T(T) denotes the distribution of the relaxation times.

Let

$$\phi \equiv \int_0^\infty f_T(T) dT = M(0).$$ (2)

In general, parameter φ corresponding to the data at t = 0 is expected to be unity since f_T is a density function. However, in this article, we consider a general case where the measured data are the Laplace transform of a scaled form of the underlying density function. For example, in the study of fluids in porous media using NMR, the amplitude f_T(T) at any given T is proportional to the number of protons relaxing at that rate. In this case, the parameter φ corresponds to the porosity of the porous medium [1]. The porosity is a critical parameter required for estimation of petro-physical properties such as permeability [1–3]. Furthermore, the estimation of the porosity is a necessary step in a new family of techniques that is being developed by the authors that allows the estimation of important parameters directly from the measurements [4–6].

The conventional approach to determine the porosity consists of estimating f_T(T) and its area using the inverse Laplace transform (ILT). However, it is well-known that the ILT is an ill-conditioned problem [7, 8] having a large sensitivity to noise in the measured data. Often, regularization or prior information about the expected density function is incorporated to make the problem better conditioned. For example, the classical approach to the problem involves choosing the "smoothest" solution f_T(T) that fits the data. This smooth solution is often estimated by minimization of a cost function Q with respect to the underlying f [9],

$$Q = \|M - K f\|^2 + \alpha \|f\|^2,$$ (3)

where M is the vector representation of the measured data, K is the matrix of the discretized kernel e^{-t/T}, and f is the discretized version of the underlying density function f_T(T). There are multiple ways to select the regularization parameter α [10–13] and each leads to a different solution f_T(T), all of which provide reasonable fits to the data.

In this paper we describe an alternative procedure to find an estimate of the porosity that does not require the prior estimation of the f_T(T). In addition, we provide equations for the uncertainty in the porosity due to the noise in the data and compare the performance of this approach with the traditional ILT method.

2. BASIC FORMULATION
Consider the NMR relaxation measurements in the √T domain M_y(y) = M(y^2) and its Fourier transform

$$\tilde{M}_y(k) = \int_{-\infty}^{\infty} M_y(y) e^{-i2\pi y^2} dy = 2 \int_0^{\infty} M_y(y) \cos(2\pi ky) dy,$$ (4)

where the second equality comes from the fact that M_y(y) is real and even [14, Eq. 2-21]. From eqns. (1) and (4) and exchanging the order of integration, we obtain

$$\tilde{M}_y(k) = \int_0^\infty f_T(T) \int_{-\infty}^{\infty} e^{-y^2/2T} e^{-i2\pi y^2} dy dT.$$ (5)

The Fourier transform of a Gaussian function is well known and is given by

$$\int_{-\infty}^{\infty} e^{-at^2} e^{-i2\pi bt} dt = \sqrt{\frac{\pi}{c}} e^{-\frac{b^2}{c}}.$$ (6)
Thus
\[ \hat{M}_y(k) = \int_0^\infty f_T(T) \sqrt{\pi T} e^{-\pi^2 k^2 T} dT. \] (7)

By integrating both sides of expression (7) over the positive frequencies we get
\[ \int_0^\infty \hat{M}_y(k) dk = \int_0^\infty f_T(T) \sqrt{\pi T} \int_0^\infty e^{-\pi^2 k^2 T} dk dT. \] (8)

The definition of the Gamma function [15, Eq. 6.1.1] can help evaluate the inner integral on the right-hand-side, yielding
\[ \phi = 2 \int_0^\infty \hat{M}_y(k) dk. \] (9)

Thus the porosity is given by the area under the Fourier transform of the data in the \( \sqrt{T} \) domain.

2.1. Uncertainty analysis

In order to determine the uncertainty of the porosity estimate we approximate the Fourier transform in (4) using trapezoidal rule,
\[ \hat{M}_y(k) = 2 \sum_{j=1}^{N} \cos(2\pi k y_j) M_y(y_j) \Delta_j \] (10)
where \( \Delta_1 = (y_2 - y_1)/2, \Delta_N = (y_N - y_{N-1})/2, \) and \( \Delta_j = (y_{j+1} - y_{j-1})/2, j = 2, \ldots, N - 1. \)

We assume that the covariance matrix of \( M_y(y) \) for \( y > 0 \) is given by the variance \( \sigma^2_\epsilon \) of the noise, \( \Sigma_{M_y} = \sigma^2_\epsilon I. \) Therefore, the covariance matrix corresponding to \( \hat{M}_y(k) \) is
\[ \Sigma_{\hat{M}_y}(n, m) = 4\sigma^2_\epsilon \sum \Delta_i^2 \cos(2\pi k_n y_i) \cos(2\pi k_m y_i). \] (11)

Similarly, the uncertainty in the porosity calculation is obtained by approximating the integral in (9):
\[ \tilde{\phi} = 2 \sum \gamma_l \hat{M}_y(k_l) \] (12)
where the weights \( \gamma_l \) are defined as in (10).

The variance of the estimated porosity is then given by
\[ \sigma^2_\phi = 4 \begin{bmatrix} \gamma_1 & \cdots & \gamma_{N_k} \end{bmatrix} \Sigma_{\hat{M}_y} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{N_k} \end{bmatrix}. \] (13)

To reduce the effect of noise in the estimated porosity, we follow the following steps. First, a simple non-negative least squares estimate of \( f_T(T) \) is found by minimizing \( Q \) in (3) assuming \( \alpha = 0. \) Using the least squares estimate of \( f_T(T), \) a fit to the data in \( \sqrt{T} \) domain and its Fourier transform \( M_y(y) \) are computed. Next, \( M_y(y) \) is low-pass filtered with filter-width being dictated by the estimated variance in (11).

![Fig. 1: Models from which synthetic data are simulated. Models A-D where chosen to have different shapes and span different regions of the \( T \) space.](image)

![Fig. 2: Example of noisy measurements with a noise level of 0.02 (left) and 0.2 (right).](image)

2.2. Simulation results

In this section we compare the performance of the proposed technique for the calculation of the porosity with the traditional ILT approach. The performance measure is the root mean square error,
\[ \text{rmse} = \sqrt{\langle (\hat{\phi} - \phi)^2 \rangle} \]
where \( \hat{\phi} \) is the estimated porosity and \( \phi \) is the true value.

We considered the four distributions shown in Figure 1. The porosity for all four distributions was chosen to be unity. Noiseless data was simulated using (1) with parameters shown in Table 1. These data were corrupted with additive white Gaussian noise at two different noise levels of \( \sigma_\epsilon = 0.02 \) and \( \sigma_\epsilon = 0.2 \) (see Figure 2) representing laboratory and field data, respectively. For reference we have...
Table 1: Parameters used to simulate data for models A-D

<table>
<thead>
<tr>
<th>Model Number</th>
<th>t_E (μs)</th>
<th>N</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>800</td>
<td>8000</td>
</tr>
<tr>
<td>B</td>
<td>2000</td>
<td>6000</td>
</tr>
<tr>
<td>C</td>
<td>200</td>
<td>2000</td>
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<tr>
<td>D</td>
<td>500</td>
<td>5000</td>
</tr>
</tbody>
</table>

Fig. 3: Estimated T distributions for 10 realizations of noise corresponding to model A. The plot shows the artifacts at the beginning of the distribution that cause the bias in the porosity.

included the estimates of the porosity obtained from solving the regularized ILT with α = 1 and α = 100 for low and high noise level, respectively. These are typical values of α for analyzing data at these SNRs.

Tables 3a and 3b show the mean μ, standard deviation σ, and rmse (in %) of the porosity estimates from 500 realizations of random noise for the two noise levels and for each model. For most cases the new approach provides estimates that have less bias and lower standard deviation in comparison to the traditional technique using regularized ILT.

The bias in the porosity calculated by the ILT method is usually caused by artifacts of the inversion as shown in Figure 3. These artifacts increase the total area of the T distribution and, therefore, increase the porosity.

The only exception is model C where the T distribution f_T(T) starts at values of T less than the echo spacing t_E (top plot in Figure 4). Since for time t = t_E the kernel in the forward model in (1) is almost zero for T < t_E (middle plot in Figure 4) the left tail of f_T(T) for T < t_E is not measured resulting in an amplitude M(t_E) that is less than the porosity (bottom plot in Figure 4).

3. CONCLUSIONS

This paper proposes a new approach for the estimation of the porosity which is a critical parameter of interest in analysis of NMR data in porous media. The new approach is based on the Fourier transform of the data in √T domain. It also provides uncertainty estimates of the porosity as a function of the standard deviation of noise in the measured data. The performance of this approach is compared to the traditional inverse Laplace transform on simulated data. It was found that this new approach provides estimates with less bias and smaller variance as long as there are no relaxation components on the order of the echo spacing t_E.

4. REFERENCES


### Table 2: Mean, standard deviation, and rmse of the estimated porosity for models A-D

<table>
<thead>
<tr>
<th>Model</th>
<th>ILT</th>
<th>New Approach</th>
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<tr>
<td></td>
<td>μ ± σ</td>
<td>rmse (%)</td>
</tr>
<tr>
<td>A</td>
<td>1.01 ± 0.009</td>
<td>1.2</td>
</tr>
<tr>
<td>B</td>
<td>1.01 ± 0.01</td>
<td>1.2</td>
</tr>
<tr>
<td>C</td>
<td>0.98 ± 0.017</td>
<td>2.7</td>
</tr>
<tr>
<td>D</td>
<td>1.01 ± 0.009</td>
<td>1.1</td>
</tr>
</tbody>
</table>

\[(a) \sigma_c = 0.02\]

<table>
<thead>
<tr>
<th>Model</th>
<th>ILT</th>
<th>New Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>μ ± σ</td>
<td>rmse (%)</td>
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<tr>
<td>A</td>
<td>1.04 ± 0.026</td>
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<tr>
<td>B</td>
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<tr>
<td>C</td>
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<tr>
<td>D</td>
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<td>4.4</td>
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</table>

\[(b) \sigma_c = 0.2\]

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