A NEW STOCHASTIC IMAGE MODEL BASED ON MARKOV RANDOM FIELDS AND ITS APPLICATION TO TEXTURE MODELING

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ABSTRACT

Stochastic image modeling based on conventional Markov random fields is extensively discussed in the literature. A new stochastic image model based on Markov random fields is introduced in this paper which overcomes the shortcomings of the conventional models easing the computation of the joint density function of images. As an application, this model is used to generate texture patterns. The lower computational complexity and easily controllable parameters of the model makes it a more useful model as compared to the conventional Markov random field-based models.

Index Terms— Stochastic image models, Markov random field, image joint density function, texture modeling

1. INTRODUCTION

Markov Random Fields (MRFs) are a well-known class of random fields that are used for stochastic modeling of 2-D spatial interactions among random variables in a lattice or in an image. Using MRFs in image processing applications such as image restoration, segmentation, and texture analysis have been thoroughly studied in the literature [1-3]. The Ising model [4] is one of the first proposed MRF-based models reflecting the interactions among pixels in 2-D lattices or images. Gibbs Random Fields (GRFs) constitute another class of random fields that their configuration obeys the Gibbs distribution. From the equivalency of MRFs and GRFs as per the theorem established by Hammersley and Clifford [5], the utilization of MRF for image modeling has become practical in image processing since the joint density of the lattice or image can be written as a Gibbs distribution. The joint density function is required in different image processing applications; however, in practice, deriving a tractable function for the joint density is a challenging task without placing the causality assumption on MRFs. The first definitions of causal (also known as unilateral) MRFs were introduced by Abend [6], Besag [7] and Pickard [8]. Using the causality assumption makes the joint density more tractable while still computing the conditional density functions as part of the causal definition of MRFs remains practically non-feasible because of the dimensionality problem. The non-symmetry property also poses difficulties in practical implementation of a unilateral model.

In this paper, first the two existing models based on the Gibbs distribution and the unilateral MRF are mentioned. Then, a new model is derived in order to overcome the shortcomings of these two models. As an application of the introduced model, a texture modeling framework is presented. Note that the terms lattice and image are used interchangeably in the paper.

In section 2, MRF and its unilateral definition is discussed. In section 3, our new model is presented. Section 4 presents a framework for texture generation as an application. Experimental results are presented in section 5 and in section 6 the paper is concluded.

2. MRF, GRF AND UNILATERAL MRF

\( X = \{X_{1,1}, \ldots, X_{m,n}\} \) is a family of random variables defined on a rectangular two-dimensional finite non-toroidal lattice of \( m \times n \) sites \( S = \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\} \). The sites correspond to the location where each random variable resides. Each random variable \( X_{i,j} \) in this lattice can take a value \( x_{i,j} \) in \( \mathcal{L} = \{1, \ldots, L\} \). \( X_{i,j} = x_{i,j} \) means that the random variable \( X_{i,j} \) takes the realization \( x_{i,j} \) that is a number in \( \mathcal{L} \). Consequently, \( (X_{1,1} = x_{1,1}, \ldots, X_{m,n} = x_{m,n}) \) denotes the joint event or a certain configuration equivalent to \( P(X = x) \), abbreviated by \( P(X) \), having \( L^{m\times n} \) distinct realizations. The neighborhood system on lattice \( S \) is defined as follows: Given a neighborhood \( N_{i,j} = \{i',j' \in S \mid \text{dist}(\text{site}(i',j'), \text{site}(i,j)) < d, (i',j') \neq (i,j)\} \), random field \( X \) is defined to be MRF if:

\[
P(X) > 0 \quad \forall X \in \mathcal{X} \tag{1}
\]

\[
P(x_{i,j} | X_{S-(i,j)}) = P(x_{i,j} | X_{N_{i,j}}) \tag{2}
\]

where \( S-(i,j) \) is the set difference, and \( x_{N_{i,j}} \) consists of the sites in \( N_{i,j} \). The first property is called positivity and the second one is known as Markovianity.

A Gibbs distribution takes the following form:

\[
P(X) = \frac{1}{Z} \exp(-U(X)) \tag{3}
\]
where \( Z \) is a normalizing constant called the partition function, \( \Omega \) is the sample space, and \( U \) is an energy function over a number of clique configurations \( C \) defined as:

\[
U(\mathbf{X}) = \sum_c V_c(\mathbf{X})
\]

with \( V_c(\mathbf{X}) \) denoting the clique potentials over all possible clique configurations.

Abend [6] also proved that

\[
p(x_{i,j} | \mathbf{X} - \{x_{i,j}\}) =
\]

\[
p(x_{i,j} | x_{i-1,j}, x_{i-1,j+1}, x_{i,j-1}, x_{i,j+1}, x_{i+1,j-1}, x_{i+1,j})
\]

As per Equation (9), the resulting neighboring configuration is shown in Fig. 3. Let us now examine the joint density function derived for a lattice based on the Gibbs distribution as indicated in Equation (3) and the unilateral MRF as indicated in Equation (8). As can be seen, the computational complexity of the Gibbs distribution is of exponential order because the normalization function in the denominator makes its computation non-feasible. On the other hand, the density function of the unilateral MRF possesses 3-D joint density terms that cannot be accurately computed due to the limited number of pixels in an image. In addition to these shortcomings, the model is non-symmetric as per Equation (3) with its spatial configuration appearing in Fig. 3. Generating a symmetric MRF was presented in [10] where based on a proposed causal Markov Mesh Random Field (MMRF), it was proved that the resulting MRF was symmetric. In what follows, we present a way to derive a symmetric and tractable model.

### 3. NEW MODEL FOR IMAGE JOINT DENSITY

To overcome the above mentioned shortcomings of MRF-based models, a new model is introduced in this section. Let us begin by defining a unilateral MRF from the right corner of a lattice.

\[
p(x_{i,j} | \mathbf{Z}_{m,n}^{i,j}) = p(x_{i,j} | x_{i-1,j}, x_{i,j-1})
\]

where \( \mathbf{Z}_{m,n}^{i,j} \) is the gray area shown in Fig. 2 and the definition is written based on order two.

In [6], Abend proved that based on the unilateral MRF definition the joint density of the lattice can be derived as follows:

\[
p(\mathbf{X}) = \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_{i,j} | x_{i-1,j}, x_{i,j-1})
\]

Based on the lattice shown in Fig. 4, the unilateral definition of MRF from the top-right corner can be written as:

\[
p(x_{i,j} | \mathbf{Z}_{m,n}^{i,j}) = p(x_{i,j} | x_{i-1,j}, x_{i,j+1})
\]

Fig. 1. Second order neighboring system and cliques: (a) single-site clique, (b)-(e) pair-site cliques.

Fig. 2. Spatial configuration of the pixels in a Unilateral MRF.

Random field \( \mathbf{X} \) is defined to be a unilateral MRF with the neighborhood \( U_{i,j} \) on a rectangular lattice if:

\[
p(x_{i,j} | \mathbf{Z}_{m,n}^{i,j}) = p(x_{i,j} | x_{i-1,j}, x_{i,j-1})
\]

where \( \mathbf{Z}_{m,n}^{i,j} \) is the gray area shown in Fig. 2 and the definition is written based on order two.

In [6], Abend proved that based on the unilateral MRF definition the joint density of the lattice can be derived as follows:

\[
p(\mathbf{X}) = \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_{i,j} | x_{i-1,j}, x_{i,j-1})
\]
And as a result, we can write
\[ p(X) = \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_{i,j}|x_{i-1,j}, x_{i,j+1}) \]  
(11)

By multiplying Equations (8) and (11), we get
\[ p(X) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left[ p(x_{i,j}|x_{i-1,j}, x_{i,j+1})p(x_{i-1,j}, x_{i,j}) \right]^{0.5} \]  
(12)

Both the practical aspect and the mathematical constraints are considered here to obtain this equation. Furthermore, by expanding the conditional probability in the above equation, we obtain
\[ p(X) = \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_{i,j})p(x_{i-1,j}, x_{i,j+1})p(x_{i-1,j+1}, x_{i,j})^{0.5} \]  
(13)

Adding the cell properties of Pickard random fields [8] to this model allows this equation to become tractable from the implementation standpoint. Thus by assuming that
\[ x_{i,j-1} \perp x_{i-1,j} | x_{i,j} & \quad x_{i,j+1} \perp x_{i-1,j} | x_{i,j} \quad (14) \]

Equation (13) can be rewritten in the following form:
\[ p(X) = \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_{i,j})p(x_{i-1,j}, x_{i,j+1})p(x_{i-1,j+1}, x_{i,j})^{2} \]  
(15)

Consequently, based on the derived model in Equation (15), we get
\[ p(x_{i,j}|x - \{x_{i,j}\}) = p(x_{i,j}|x_{i-1,j}, x_{i,j-1}, x_{i,j+1}, x_{i+1,j}) \]  
(16)

The model described by Equation (15) has two main advantages as compared to the conventional MRF models. First, it is tractable since it has at most 2-D joint density terms that can be easily estimated using the histogram approximation method. It is worth noting that the 2-D joint density functions at different directions can be easily computed, for example, the joint density of a pixel to the right neighboring pixel. Second, it is symmetric since the dependency of a pixel to its right and left pixels are considered together in the equation which makes it more usable in image processing applications.

4. TEXTURE MODELING AS AN APPLICATION

Without loss of generality, the framework derived here is for a 2-level texture; however, it can be easily extended to higher levels. One can write:
\[ p(x_{r,c}|x - \{x_{r,c}\}) = \frac{p(x_{r,c}, x - \{x_{r,c}\})}{p(x - \{x_{r,c}\})} = \frac{p(X)}{\sum_{x_{r,c}} p(X)} = \]  

\[ p(x_{r,c} = l_{r}|x - \{x_{r,c}\}) = \frac{\prod_{i=1}^{m} \prod_{j=1}^{n} p(l_{r}, x_{i,j})p(l_{r}, x_{i+1,j})^{0.5} \prod_{i=1}^{m} \prod_{j=1}^{n} p(x_{i-1,j}, x_{i,j})^{0.5}}{\sum_{x_{r,c}} p(x_{i,j}, x_{i,j+1})p(x_{i-1,j}, x_{i,j})^{0.5}} \]  
(17)

By further expanding Equation (17) for a two-level model, Equation (18) is resulted.

In 2-level texture modeling, \( x_{r,c} = \{l_{r}, l_{2}\} = \{0, 1\} \).

Naturally, for higher level texture modeling, this equation needs to be modified accordingly.
This equation is used here to serve as the framework to generate texture patterns. The joint distribution of two neighboring pixels for an $L$-level image is an $L \times L$ matrix incorporating $L \times L$ parameters. For generating a 2-level texture, there are 4 parameters. Let us denote these parameters by $h_{11} = p(l_1, l_1)$, $h_{12} = p(l_1, l_2)$, $h_{21} = p(l_2, l_1)$ and $h_{22} = p(l_2, l_2)$ reflecting the joint probability of a pixel and its immediate neighbor (refer to Figs. 1(b) and (c)). The 2 joint density matrices can be considered to be texture parameters. Let us write them compactly in matrix form as $h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$. For an $L$-level texture, there are 2 matrices leading to $2 \times L^2$ parameters for texture generation.

5. EXPERIMENTAL RESULTS

The widely used method for generating MRF-based texture patterns consists of using MRF parameters as a control mechanism for their shape [11]. Here, as a new approach, we utilize the joint entropy terms instead of the MRF parameters to generate texture patterns. To generate texture patterns based on our model, first a texture pattern is initialized by randomly choosing levels $l_1$ and $l_2$. Then, the texture pattern is sampled by the Gibbs sampler [9] to select the minimal energy among various possible configurations. A pixel in the texture is replaced by $l'$ with probability $p(x_{r,c} = l | x - \{x_{r,c}\})$. This process is iterated for all the pixels in the image until reaching the equilibrium which can be regarded as the minimal energy of the texture pattern.

Figures 5(a) and (b) show the generated texture patterns using a 2-layer model with $h = \begin{pmatrix} 0.490 & 0.010 \\ 0.010 & 0.490 \end{pmatrix}$ and $h = \begin{pmatrix} 0.475 & 0.025 \\ 0.025 & 0.475 \end{pmatrix}$ for both the horizontal and vertical pair-wise neighboring sites, respectively. Figures 5(c) and (d) show the generated texture patterns using a 3-layer model with $h = \begin{pmatrix} 0.300 & 0.003 & 0.0017 \\ 0.0016 & 0.300 & 0.0017 \\ 0.0016 & 0.0017 & 0.300 \end{pmatrix}$ and $h = \begin{pmatrix} 0.3100 & 0.0011 & 0.0011 \\ 0.0012 & 0.3100 & 0.0012 \\ 0.0011 & 0.0012 & 0.3100 \end{pmatrix}$ for both the horizontal and vertical pair-wise neighboring sites, respectively. In Figs. 5(e) and (f), a 64-level and a 128-level texture patterns are generated. The first one is based on a $64 \times 64$ joint distribution matrix with the diagonal elements of 0.99/64 and the off-diagonal elements of 0.1/(64 × 63). The 128-level texture pattern is generated based on a $128 \times 128$ joint density matrix with the diagonal elements of 0.99/128 and the off-diagonal elements of 0.1/(128 × 127). The diagonal elements of the model in fact influence the individual contribution of each level to a texture pattern while the off-diagonal elements influence the combined contribution of the levels. Placing more weight on diagonal elements produces larger areas at each level (refer to Fig. 5(b) and (d)) and lower weight on diagonal elements produces smaller areas (refer to Fig. 5(b) and (d)). Higher level texture patterns are shown here to confirm that the proposed model is tractable even in the presence of more levels. This is in fact a shortcoming of the conventional MRF models by becoming non-tractable when the number of levels goes high.

6. CONCLUSION

This paper has provided a new stochastic model for images based on MRFs. It was shown that the proposed model is symmetric and comprises the product of at most 2-D joint density functions making it tractable and thus a useful tool in image processing applications. To demonstrate its usefulness, an application consisting of texture generation was also discussed by considering the elements of the joint density function as texture parameters controlling the shape of texture patterns. The main attribute of the introduced model is that its parameters can be estimated in a computationally feasible manner. This model can be considered for various applications in image processing.

7. REFERENCES