CHARACTERIZATION OF THREE-DIMENSIONAL DATA WITH MULTIDIMENSIONAL DEFORMABLE MODELS BASED ON B-SPLINES IN THE FOURIER DOMAIN


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ABSTRACT
This paper presents the formulation and implementation of multidimensional deformable models in the frequency domain. The models are defined by finite element discretization, using B-splines as shape function. The frequency-based formulation allows us to obtain an efficient iterative system in matrix form, suitable for models of any dimension. The aim of this approach is the use of these models for fast segmentation and motion tracking of three-dimensional objects with non-rigid motion or deformation from three-dimensional datasets. To show the results of the process, a two-dimensional model is applied for the characterization of topographic data.

Index Terms— multidimensional deformable model, B-spline, frequency domain, three-dimensional segmentation, motion tracking

1. INTRODUCTION
Active contours [1] have been used in recent years for many applications in segmentation and tracking of moving objects from images sequences [2, 3]. The formulation of the contours in the frequency domain [4], reduces the computational load of the iterative process. The extrapolation of active contours to active-meshes, has also been used in several applications, although the models used in them are defined in the space domain [5, 6, 7].

This paper presents a formulation of multidimensional deformable models from a $d$-dimensional generalization of [1], using B-splines as shape function and a frequency-based formulation similar to [4]. This allows to apply the advantages of the frequency formulation to multidimensional data. The paper is organized as follows: In Section 2 we describe the formulation of the models, Section 3 details the implementation, and finally, in Section 4 the results of the iterative process applied to topographic data are shown.

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2. MULTIDIMENSIONAL DEFORMABLE MODELS
2.1. Parametrical deformable models
A dynamic deformable model is a generalization of the active contours described by Liang et al. [1]. Thus, an active deformable model is defined as a parametric curve, surface or hypersurface in the $d$-dimensional space $\mathbb{R}^d$,

\[
\mathbf{v}(s, t) = [v_1(s, t), v_2(s, t), \ldots, v_d(s, t)]^T,
\]

where $s = [s_1, \ldots, s_e]$ with $s_j \in [0, L_j]$ and $e \leq d - 1$ is the vector of the parametric variables of the space domain, $t$ represents the time and $v_i(s, t)$ is the $i$th coordinate function. A deformable model is closed when satisfies for all coordinate functions, $v_i(s)\big|_{s_j=0} = v_i(s)\big|_{s_j=L_j} \forall i \forall j$. Otherwise, the model is open for the parametric variables $s_j$ that do not fulfill this condition.

As a first approach, we consider the model as a static process, $\mathbf{v} = \mathbf{v}(s)$. The model is governed by an energy functional $E(\mathbf{v}) = S(\mathbf{v}) + P(\mathbf{v})$ [8]. The first term is the internal deformation energy and characterizes the elastic deformation of a flexible model,

\[
S(\mathbf{v}) = \frac{1}{2} \sum_{i=1}^{d} \left( \int_{\Omega} \alpha_i(s) \| \nabla v_i(s) \|^2 + \beta_i(s) |\Delta v_i(s)|^2 \ ds \right)
\]

where $\Omega := [0, L_1] \times \ldots \times [0, L_e]$ is the integration domain, $\nabla$ and $\Delta$ represent the gradient and Laplacian operators respectively, and the non-negative parameters $\alpha_i(s)$ and $\beta_i(s)$ control the elasticity (or "tension") and the rigidity in any coordinate $s$ of the model. The term $P(\mathbf{v})$ contains the rest of energies applied to the model. This term typically includes energies obtained from external sources, such as a set of pictures or multidimensional objects. Different restrictions can also be included as nonlinear internal forces defined inside the model.

Since the internal and external forces are applied together, the equilibrium is reached when the energy functional is minimum. According to the variational calculus [9], the model $\mathbf{v}(s)$ that minimizes $E(\mathbf{v})$ must
satisfy the Euler-Lagrange (E-L) equations [8], which
genralized to the multidimensional case produce a set of
differential equations (PDE) synthesized in
the following vector equation [10],
\[-\nabla \cdot \left( \alpha(s) \nabla v(s) \right) + \Delta \left( \beta(s) \Delta v(s) \right) = q(v(s)), \quad (3)\]
where \(\nabla\) represents the divergence operator and the
term \(q(v(s)) = -\nabla v \cdot P(v(s)) + f(v(s)) \in \mathbb{R}^n\) represents
the external forces and restrictions on the model. \(\nabla v \cdot P\)
is the gradient operator regarding to the coordinates \(v_i\)
of the potential function \(P\). \(f\) provides the restrictions
for each coordinate function.

In order to define the minimization process as a
dynamical system, \(v \equiv v(s, t)\), it is common to add a mass
density \(\mu(s)\) and a damping density \(\gamma(s)\) to (3),
\[
\mu(s) \partial_t v(s, t) + \gamma(s) \partial_t v(s, t) -
\nabla \cdot \left( \alpha(s) \nabla v(s, t) \right) + \Delta \left( \beta(s) \Delta v(s, t) \right) = q(v(s, t)), \quad (4)
\]
where \(\partial_t\) and \(\partial_{tt}\) denote respectively first and second
partial derivative with respect to time. The first two
terms represent the inertial forces due to the mass and
damping forces. The next two terms represent the
internal stretching and bending deformation forces. By
applying the operators \(\nabla, \Delta\) and \(\cdot\), and assuming that
each coordinate function of \(f\) can be determined inde-
pendently of the rest of coordinates, a system of \(d\) de-
coupled PDE is obtained. Each of these equations can
be solved independently of the rest by means of numerical
algorithms.

2.2. B-splines formulation in the spatial domain

Practical implementations of deformable models are
usually performed in discrete time and space. In our
model, the temporal discretization is using a per-
iodic sampling. For the spatial discretization, the
deformable model will be represented by finite elements
[11]. For each coordinate function \(v_i\), the parametric
domain \(0 \leq s_i \leq L_i\) is divided into \(N_i\) intervals. This way
the model can be expressed as the union of \(N_1 \cdots N_n\)
elements,
\[
v_i(s, t) = \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_n=0}^{N_n-1} v_i^{n_1, \ldots, n_n}(s, t), \quad (5)
\]
Each element \(v_i^{n_1, \ldots, n_n}\), is represented geometrically
using shape functions \(N_i^{n_1, \ldots, n_n}(s)\) and nodal variables
\(u_i^{n_1, \ldots, n_n}(s, t)\). Assuming a closed model, identical
shape functions for all segments and constant param-
eters within the segment, each finite element can be expressed by
\(v_i^{n_1, \ldots, n_n}(s, t) = N_i(s) u_i^{n_1, \ldots, n_n}(t)\).

The uni-dimensional shape functions commonly
used in literature [1] are the Hermite polynomials,
B-splines and finite differences. In the case of mul-
dimensional deformable models, it is necessary to
decompose the multidimensional polynomial in
the product of separable one-dimensional polynomials,
\(N(s) = N_1(s_1) \cdots N_n(s_n)\). This is possible only for B-
splines and finite differences. In this article we present
the case of the implementation of a deformable models
with B-splines.

The use of finite element formulation leads to an it-
terative system into a matrix form. This allows the effi-
cient coupling of deformable model to external data. By
applying the Garlekin’s method to the system of PDE
similar to [1], we transform the E-L equations of motion
in a second-order ordinary differential equations,
\[
M \ddot{u}_i(t) + C \dot{u}_i(t) + K u_i(t) = q_i(t), \quad (6)
\]
where \(M\) is the mass matrix, \(C\) is the damping matrix, \(K\)
is the stiffness matrix and \(q_i\) is the external forces vector.
Note that the use of (6) is only possible if the nodes \(u_i\)
are defined as a column vector\(^2\).

Using multidimensional B-splines as shape functions,
each \(v(s, t)\) can be constructed as a weighted sum of
basis functions [12]. Each of them can be calculated
as the product of one-dimensional B-splines,
\[
v(s, t) = \sum_{\pi} B_{n_1}(s_1) \cdots B_{n_n}(s_n) \tilde{u}(t), \quad (7)
\]
where \(\pi = [n_1, \ldots, n_n]\) are the indexes of the segments
in each dimension and \(\tilde{u}(t)\) are the nodal variables.

The model domain is divided into a number of small
subdomains [1]. Each subdomain can be considered as
an elementary deformable model. The global matrices
for the entire model (6) can be calculated by assembling
the submatrices of these subdomains. Note that \(M, C\)
and \(K\) have dimensions of \((N_1 \cdots N_n) \times (N_1 \cdots N_n)\)
that fit with the size of \((N_1 \cdots N_n) \times 1\) of the rearranged
vector \(u(t)\). \(M\) and \(C\) satisfy \(M = mF + C = CF\), where
\(m\) and \(c\) represent the mass and stiffness of the model and
\(F\) is the shape matrix. The time is discretized, \(t = \xi \Delta t\),
being \(\Delta t\) the time step and \(\xi \in \mathbb{N}\) the iteration index\(^3\).

The time derivatives of \(u(t)\) are replaced by their dis-
crete approximations [10]. Thus, (6) can be written as,
\[
\left( \begin{bmatrix} m \Delta t^2 + c \Delta t \end{bmatrix} F + K \right) u_\xi = \begin{bmatrix} 2m \Delta t^2 + c \Delta t \end{bmatrix} F u_{\xi-1} + \begin{bmatrix} -m \Delta t^2 \end{bmatrix} F u_{\xi-2} + q_{\xi-1}, \quad (8)
\]
being \(F\) and \(K\) nested circulant matrices. Also for the
multidimensional case and similarly to active contours
\(^1\)To simplify the nomenclature, subscript \(i\), which indicates the co-
ordinate function, is hidden from this point.
\(^2\)For deformable models of two or more dimensions, the multi-
dimensional array of nodes \(u^n\) must be reshaped to a column vector \(u\).
\(^3\)In the following, the notation \(u_\xi(\xi \Delta t) = u_\xi\) is used.

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[4], the result of multiplying a nested circular matrix and a vector is equivalent to a circular convolution between two multidimensional signals,
\[ \eta^{-1} (\eta f + k) @ u_\xi = a_1 f @ u_{\xi-1} + a_2 f @ u_{\xi-2} + \eta^{-1} q_{\xi-1}, \]
where @ indicates \( \eta \)-dimensional circular convolution of arrays of size \( N_1 \times \cdots \times N_\eta \), implies the multiplication of their DFT's. This allows us to isolate \( u_\xi \),
\[ \hat{u}_\xi = \hat{h} \left( a_1 \hat{u}_{\xi-1} + a_2 \hat{u}_{\xi-2} + (\eta \hat{f})^{-1} \hat{q}_{\xi-1} \right), \]
where \( \hat{u} \) and \( \hat{q} \) are the DFT's of their respective spatial sequences. \( \hat{h} = 1/(1 + k/(\eta \hat{f})) \) is an \( \eta \)-dimensional low-pass filter, inverse of the high-pass filter \( k \) [10]. Equation (10) provides an efficient formulation for the tracking process with deformable models. Appendix A shows the analytical expressions of \( f \) and \( k \).

2.3. Formulation in the frequency domain
The discrete spatial domain \( \mathbf{w} = [\omega_1, \ldots, \omega_\eta] \). The \( \eta \)-dimensional discrete Fourier transform of the circular convolution of two periodical sequences⁴, implies the multiplication of their DFT's. This allows us to isolate \( u_\xi \),
\[ \hat{u}_\xi = \hat{h} \left( a_1 \hat{u}_{\xi-1} + a_2 \hat{u}_{\xi-2} + (\eta \hat{f})^{-1} \hat{q}_{\xi-1} \right), \]
where \( \hat{u} \) and \( \hat{q} \) are the DFT's of their respective spatial sequences. \( \hat{h} = 1/(1 + k/(\eta \hat{f})) \) is an \( \eta \)-dimensional low-pass filter, inverse of the high-pass filter \( k \) [10]. Equation (10) provides an efficient formulation for the tracking process with deformable models. Appendix A shows the analytical expressions of \( f \) and \( k \).

3. PRACTICAL IMPLEMENTATION
To check the validity of this approach, we apply a 2D deformable model to different topographical contours. Note that although for this experiment the data are static, the dynamic characteristics of the system allow the model to track data changing over the time.

The model is defined by three coordinate functions \([u_{x,\xi}, u_{y,\xi}, u_{z,\xi}]^T\) determined by two discrete parametric variables \([n_1, n_2]\). \( u_x \) and \( u_y \) are set constant over the time with an equispaced mesh. In contrast, \( u_z,\xi \) is governed by (10): external forces fit the model to the data whereas internal forces allow a smooth adjustment while reducing the effects of noise and other artifacts.

As a requirement of the proposed formulation, the deformable model has to be closed. Since the topographic data require an open model, during the filtering process it is necessary to add extensions that emulate a closed model at the ends of the parametric domains [4]. These extensions are recalculated with each iteration and removed from the final result of the process.

As in the case of one-dimensional models, the tracking of the model to data is performed efficiently using the gradient operator as external forces. Thus, topographic height data are mapped into a 3D volume. Then, the gradient of the volume is calculated. Since \( u_x \) and \( u_y \) are fixed, we use only the \( z \) component of the gradient as external forces \( q_{z,\xi} \) of the iterative process.

The choice of parameters of the model depends on the characteristics of the data. Data with large variation involve high \( \alpha \) and small \( \beta \) values (model with more elasticity and less rigidity). \( \gamma \) and \( \eta \) parameters influence the convergence rate [4]. Note that a faster convergence implies however a greater oscillation of the nodes in equilibrium (and therefore a bigger final error).

4. VALIDATION OF THE MODEL AND RESULTS
The iterative process (10) has been applied to two data sets: (a) Peaks, synthetic data with soft oscillations and (b) Mountain, real data with sharper variations. Both data sets have the same dimensions, \( 454 \times 319 \times 100 \) voxels. Table 1 shows the parameters used. In all experiments, the \( q_z \) coordinate of the model has been initialized to a constant value.

<table>
<thead>
<tr>
<th>Peaks</th>
<th>Mountain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0.15 )</td>
<td>( \alpha = 0.45 )</td>
</tr>
<tr>
<td>( \beta = 0.2 )</td>
<td>( \beta = 0.15 )</td>
</tr>
<tr>
<td>( \eta = 12 )</td>
<td>( \eta = 6 )</td>
</tr>
<tr>
<td>( \gamma = 0.05 )</td>
<td>( \gamma = 0.01 )</td>
</tr>
</tbody>
</table>

Table 1. Optimal parameters for the data sets.

Fig. 1 depicts 3 iterations of the deformable model with size \( N_1 = N_2 = 64 \). As can be seen, \( q_z \) adapts quickly to the data in both cases, although the adaptation of Mountain is slower due to its lower smoothness. Fig. 2 compares the average error over time for the two data sets with different model sizes. As expected, the convergence of Peaks is higher than Mountain (higher speed and lower final average error). For larger models, the adjustment process is slower but the final average error is smaller. Fig. 2 also illustrates the problem of convergence of Mountain with \( N_1 = N_2 = 32 \). Due to the sharp shape of the data and the smoothing effect of B-splines, the model is not able to fit the data.

5. CONCLUSIONS
In this paper, a new formulation for multidimensional deformable models has been described and analyzed. This approach has advantages such as compact and efficient formulation and can be applied to models of any dimension. This work provides a new perspective to analyze and design deformable models, which can be extended easily to other shape functions and customized by the appropriate choice of parameters for different applications of multi-dimensional motion tracking and segmentation. Static and dynamic three-dimensional medical images, sonar and radar data, stereoscopic image pairs and data from 3D scanners are some of the applications of the formulation here presented.

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⁴This implies a closed deformable model.
This appendix provides analytical expressions of the frequency responses of the filters $F$ and $K$ for 1 and 2 dimensions. These expressions can be obtained directly by applying the Fourier transform of the sequences $f$ and $k$ in (9),

$$F^{(1)}(\omega) = \frac{1}{2520} (1208 + 1191 \cos(\omega) + 120 \cos(2\omega) + \cos(3\omega)), \quad (11)$$

$$K^{(1)}_{\alpha}(\omega) = \frac{1}{60} (40 - 15 \cos(\omega) - 24 \cos(2\omega) - \cos(3\omega)), \quad (12)$$

$$K^{(1)}_{\beta}(\omega) = \frac{1}{3} (8 - 9 \cos(\omega) + \cos(3\omega)), \quad (13)$$

Note that the terms (11-17) are continuous. Discrete filters $f$ and $k$ can be obtained as $\hat{f} = F[k_1, \ldots, k_e] = F(\omega_1, \ldots, \omega_e)|_{\omega_j = \frac{j}{N_j}}$, with $k_j \in [0, N_j - 1]$.

### A. APPENDIX

Fig. 1. Temporal evolution with mesh size $N_1 = N_2 = 64.$

### B. REFERENCES


