ABSTRACT
We consider the task of acoustic system identification, where the input signal undergoes a memoryless nonlinear transformation before convolving with an unknown linear system. We focus on the possibility of modeling the nonlinearity with different basis functions, namely the established power series and the proposed Fourier expansion. In this work the unknown coefficients of generic basis functions are merged with the unknown linear system to obtain an equivalent multichannel structure. We use a multichannel DFT-domain algorithm for learning the underlying coefficients of both types of basis functions. We show that the Fourier modeling achieves faster convergence and better learning of the underlying nonlinearity than the polynomial basis.

Index Terms— Memoryless nonlinearity, polynomial expansion, Fourier series, multichannel algorithm

1. INTRODUCTION
Identification of an unknown system is a problem ubiquitously present in signal processing and control systems. The identification task becomes more challenging when the input signal undergoes a nonlinear transformation before convolving with the unknown system. The challenge can only be resolved if the nonlinearity is effectively modeled within the system identification framework. Nonlinear acoustic system identification is of particular relevance in applications of acoustic echo cancellation [1], and the design of digital audio effects [2] and loudspeakers [3].

Nonlinear transformations on the input signal have been typically classified under mappings with and without memory [1], both of which have very pertinent application specific utilization. The nonlinearity with memory can be modeled with the help of Volterra series [4][5], leading to algorithms with considerable complexity. High complexity becomes inevitable due to the learning of multidimensional kernels. The memoryless nonlinearity, which is the focus of this work, has been modeled traditionally in the acoustic signal processing via the power series [1][2]. Power series expansion offers a convenient mathematical model, which can be easily utilized to setup an equivalent multichannel linear structure by absorbing the coefficients of the nonlinear mapping into the unknown system. Multi-channel representations do not always promise a minimum complexity scenario, however, the identification task becomes procedurally simple. Such multichannel formulations can suffer from correlation existing between input of different channels. Practitioners of polynomial modeling, hence, augment the approach with orthogonalizing procedures, e.g., Gram-Schmidt orthogonalization [1].

In this paper, along with considering the established power series modeling in the acoustic domain, we suggest a nonlinear expansion using the truncated Fourier series. We show that Fourier series results in a multichannel algorithm, where channels are directly fed by mutually orthogonal signals. This enables us to circumvent any additional orthogonalization mechanism. We recognize the possibility of achieving similar advantage by delving into orthogonal polynomial expansions, e.g., Legendre polynomials [6]. Nevertheless, in this work we aim to highlight the core principle and advantages of using orthogonal basis by means of the Fourier series.

In Sec. 2 we present the system model and the mathematical expressions of the basis functions for the modeling of the nonlinearity. Here we also establish the plausibility of modeling via the Fourier basis by means of a curve-fitting example. In view of the aforementioned basis functions, Sec. 3 deals with the derivation of our basis-generic DFT-domain multichannel signal model, followed by the presentation of a multichannel adaptive algorithm. We present results in Sec. 4 for both types of basis functions and the conclusions of our endeavors can be found in Sec. 5.

We use nonbold lowercase letters for scalar quantities, bold lowercase for vectors, and bold uppercase for frequency-domain quantities. The superscript $H$ denotes Hermitian transposition. We use $F_M$ and $\log$ to denote the DFT matrix of size $M$ and the natural logarithm, respectively. Lowercase letters “$i$” and “$r$” are reserved for sample- and block-time indices, where $R$ is the block-shift.

2. BASIS FUNCTIONS FOR NONLINEAR EXPANSION
Figure 1 describes a noisy signal model with a memoryless nonlinear mapping. The input signal $x_t$ is nonlinearly mapped via an unknown transformation $f[\cdots]$ to give $f[x_t]$. The nonlinearly mapped input signal $f[x_t]$ is then convolved with the unknown system $w_t'$ forming the intermediate signal $d_t$, which is corrupted by the observation noise $s_t$ resulting in the observation $y_t$. Therefore, comprehensive system identification requires simultaneous learning of the unknown linear and nonlinearity parts of the system.

At first, we present the modeling of the nonlinearly mapped input signal $f[x_t]$ by means of a generic $p$-th order expansion

$$f[x_t] = \sum_{i=1}^{p} a_i \phi_i(x_t), \quad (1)$$

This work was carried out in cooperation with Nokia.
where $a_i$ and $\phi_i(\cdots)$ are $i$-th order coefficient and basis function of the nonlinear expansion, respectively. Such a basis-generic expression is very handy in our case, as we will later extend it to our multichannel algorithm without any loss of generality.

2.1. Polynomial Expansion

We consider (1) and specify the traditional power series expansion of $f[x_t]$ by defining $\phi_i(\cdots)$ as the $i$-th power of the input signal $x_t$

$$f[x_t] = \sum_{i=1}^{p} a_i x_t^i,$$  \hspace{1cm} (2)

such that $a_i$ becomes the $i$-th order coefficient of a $p$-th order polynomial. The coefficients $a_i$ can be computed numerically in the least squares sense to fit the transformation $f[x_t]$, i.e.,

$$a_i = \arg \min_{a_i} \int_{-1}^{1} \left[ f[x_t] - \sum_{i=1}^{p} a_i x_t^i \right]^2 dx_t, \forall i,$$  \hspace{1cm} (3)

assuming normalized input signal magnitudes, i.e., $x_t \in [-1, 1]$.

2.2. Fourier Expansion

By definition, the Fourier series is described as an infinite summation of sine and cosine terms. To customize the Fourier series for our application we first analyze the attributes of the nonlinearity at hand. Due to the odd symmetry of typical nonlinear mappings the cosine terms of the Fourier series can be omitted. Thus, we describe the nonlinear mapping by means of a truncated odd Fourier series

$$f[x_t] = \sum_{i=1}^{p} a_i \phi_i(x_t) = \sum_{i=1}^{p} a_i \sin(\pi i \cdot \frac{x_t}{L}),$$  \hspace{1cm} (4)

where $2 \cdot L$ denotes the fundamental period. The coefficient $a_i$ in (4) are odd Fourier coefficients in closed form and are computed via

$$a_i = \frac{1}{L} \int_{-L}^{L} f[x_t] \cdot \sin(\pi i \cdot \frac{x_t}{L}) dx_t, \forall i.$$  \hspace{1cm} (5)

2.3. Comparison of the Fitting Ability

Before moving on to embedding the aforementioned nonlinear models in a system identification algorithm, we strengthen our motivation by visually showing that similar to a polynomial series, the Fourier expansion can also effectively model the nonlinear mapping. We consider the memoryless nonlinearity $f[\cdots]$ as such that

$$f[x_t] = \left\{ \begin{array}{ll} x_t & \text{if } |x_t| \leq x_{\text{max}}, \\ -x_{\text{max}} & \text{if } x_t > x_{\text{max}}, \\ x_{\text{max}} & \text{if } x_t < -x_{\text{max}}, \end{array} \right.$$  \hspace{1cm} (6)

where $x_{\text{max}}$ is the clipping threshold. For the analysis we consider a clipping function with a clipping threshold $x_{\text{max}}$ of 0.1. Such a clipping threshold offers a very realistic nonlinear configuration for input data normalized within the range $-1 < x_t < 1$.

For our curve-fitting experiment we consider an expansion order of $p = 5$ for both types of expansions. The polynomial coefficients are computed considering (3). The Fourier coefficients are computed using (5). Here, the period $2 \cdot L$ indeed is a term to be reckoned with. Of course, the clipping function is not periodic in the true sense. Nevertheless it can be modeled as one without harm, as the input data is restricted within a specified amplitude range. In the example, we select $L$ greater than 1, as otherwise the truncated Fourier series will never be able to fit the clipping function on the fringes of the data range. The sinusoids which make up the series are all zero at the specified value of $L$, and hence our selection of $L = 1.5$.

In Fig. 2 we depict, with the help of the computed polynomial and Fourier expansion coefficients, the modeled clipping functions. It can be observed that while both forms of expansion follow the ideal curve, the Fourier model fits better to the clipping function. Thus, the Fourier expansion can rightly be considered a contender for such a modeling problem.

3. MULTICHANNEL ADAPTIVE IDENTIFICATION

Owing to the reported advantages of frequency domain adaptive filtering, both in single channel [7] and multichannel representations [8], we proceed with the formulation of our multichannel algorithm for system identification in DFT-domain.

3.1. Nonlinear Signal Model in DFT-Domain

The samples of the nonlinearly mapped input signal $f[x_t]$ can be lumped together in a block-based representation

$$\mathbf{f}_r = \{ f[x_{\tau R - M + 1}], f[x_{\tau R - M + 2}], \cdots, f[x_{\tau R}] \}^H,$$  \hspace{1cm} (7)

where $\tau$ denotes the block-index and $R$ is the block-shift. We substitute into (7) the mapping given in (1) to get

$$\mathbf{f}_r = \left\{ \sum_{i=1}^{p} \tilde{a}_i \phi_i(x_{\tau R - M + 1}), \sum_{i=1}^{p} \tilde{a}_i \phi_i(x_{\tau R - M + 2}), \cdots, \sum_{i=1}^{p} \tilde{a}_i \phi_i(x_{\tau R}) \right\}^H,$$  \hspace{1cm} (8)

which can be conveniently rearranged to obtain

$$\mathbf{f}_r = \sum_{i=1}^{p} \tilde{a}_i \phi_i(x_{\tau R - M + 1}), \phi_i(x_{\tau R - M + 2}), \cdots, \phi_i(x_{\tau R}) \}^H = \sum_{i=1}^{p} \tilde{a}_i \mathbf{x}_{\tau, i},$$  \hspace{1cm} (9)
where $x_{\tau,i} = \{\phi_i(x_{\tau,R-M+1}), \phi_i(x_{\tau,R-M+2}), \ldots, \phi_i(x_{\tau,R})\}^H$ is the $i$-th order of the block-based non-linearly mapped input signal. We attain a DFT-domain representation by first applying the DFT matrix followed by diagonalization to (9)

$$\tilde{X}_\tau = \text{diag}\{F_Mf_\tau\} = \sum_{i=1}^p a_i\text{diag}\{F_Mx_{\tau,i}\}$$

$$= \sum_{i=1}^p a_iX_{\tau,i},$$  \hspace{1cm} (10)

where $X_{\tau,i} = \text{diag}\{F_Mx_{\tau,i}\}$ represents the $i$-th order.

We now model $M - R$ non-zero coefficients of the unknown system $w_\tau = w_\tauR$ such that its DFT-domain representation

$$W_\tau = F_M\left[w_\tau^H 0_R\right]^H$$  \hspace{1cm} (11)

holds. Using (11) and an $R \times M$ projection matrix $Q^H = (0 \ I_R)$ we can express the observed signal $y_\tau$ using overlap-save as

$$Y_\tau = F_MQy_\tau = F_MQQ^HH\tilde{X}_\tauW_\tau + S_\tau,$$  \hspace{1cm} (12)

where $y_\tau = [y_{\tau,R-R+1}; y_{\tau,R-R+2}; \ldots; y_{\tau,R}]$ and $S_\tau = F_MQs_\tau$ is the DFT-domain observation noise vector. The time-domain observation noise vector $s_\tau$ is defined analogous to $y_\tau$.

### 3.2. Equivalent Multichannel System

With the substitutions $G = F_MQQ^HF_M^{-1}$ and $\tilde{X}_\tau = \sum_{i=1}^p a_iX_{\tau,i}$, (12) can be reformulated as

$$Y_\tau = G\sum_{i=1}^p a_iX_{\tau,i}W_\tau + S_\tau = \sum_{i=1}^p a_iGX_{\tau,i}W_\tau + S_\tau$$

$$= \sum_{i=1}^p a_iC_{\tau,i}W_\tau + S_\tau.$$  \hspace{1cm} (13)

In (13) the compact notation $C_{\tau,i} = GX_{\tau,i}$ represents the $i$-th order of the non-linearly mapped overlap-save constrained DFT-domain input signal. We combine the non-linear coefficients $a_i$ with the unknown system $W_\tau$ to get the multichannel representation, i.e.,

$$Y_\tau = [C_{\tau,1} C_{\tau,2} \cdots C_{\tau,p}] \begin{bmatrix} a_{11M} & \ldots & a_{1pM} \\ a_{21M} & \ldots & a_{2pM} \\ \vdots & \ddots & \vdots \\ a_{p1M} & \ldots & a_{ppM} \end{bmatrix} W_\tau + S_\tau$$

$$= [C_{\tau,1} C_{\tau,2} \cdots C_{\tau,p}] \begin{bmatrix} W_{\tau,1} \\ W_{\tau,2} \\ \vdots \\ W_{\tau,p} \end{bmatrix} + S_\tau = C_{\tau}W_\tau + S_\tau,$$  \hspace{1cm} (14)

which is also illustrated in Fig. 3. Thus the system identification task has been transformed into an equivalent multichannel problem.

### 3.3. Multichannel Adaptive Filtering

For the identification of the unknown system model that fits the data $y_\tau$, we consider an unconstrained frequency-domain adaptive algorithm of the form [7][9]

$$E_\tau = Y_\tau - C_{\tau}\tilde{W}_{\tau-1}$$

$$\tilde{W}_\tau = \tilde{W}_{\tau-1} + \mu X_{\tau}^HE_\tau,$$  \hspace{1cm} (15)

where $E_\tau$ is the error signal and $\tilde{X}_\tau^H = [X_{\tau,1} X_{\tau,2} \cdots X_{\tau,p}]^H$. The $pM \times pM$ step-size matrix $\mu_{\tau,i}$ which can provide a separate step-size factor for each channel, is described by the matrix

$$\mu_{\tau,i} = \begin{bmatrix} \mu_{\tau,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mu_{\tau,i} & 0 \\ 0 & \cdots & \cdots & \mu_{\tau,p} \end{bmatrix},$$

where $\mu_{\tau,i}$ is an $M \times M$ diagonal matrix for the $i$-th channel, with each entry denoting an individual step-size for each frequency bin:

$$\mu_{\tau,i} = \sigma_{X_{\tau,i}}^{-1}X_{\tau,i}^{-\dagger}X_{\tau,i}.$$  \hspace{1cm} (16)

The terms $\Psi_{X_{\tau,i}X_{\tau,i}}$ and $\alpha_i$ are the power spectral density estimate of the input signal $X_{\tau,i}$ and the adaptation constant in the range $0 < \alpha_i < 1$ for the $i$-th channel, respectively. The $M \times M$ diagonal matrix $\Psi_{X_{\tau,i}X_{\tau,i}}$ is computed by recursive averaging,

$$\Psi_{X_{\tau,i}X_{\tau,i}} = \gamma_i\Psi_{X_{\tau,i-1}X_{\tau,i-1}} + (1 - \gamma_i)X_{\tau,i}^H X_{\tau,i},$$  \hspace{1cm} (17)

where $\gamma_i$ is a forgetting factor in the range $0 < \gamma_i < 1$. Additional constraining [9] can be applied to the update equation in (15) in order to avoid cyclic terms in the cross-correlation $\tilde{X}_{\tau}^HE_\tau$.

### 4. RESULTS

We considered a hard clipping function with a threshold of 0.1 as the non-linearity, as depicted earlier in Fig. 2, with the sampling frequency of $f_s = 16k$Hz. The linear-to-non-linear power ratio after applying a white input signal to this nonlinearity amounted to $\sigma_x^2/\sigma_{f[x]}^2 = 5$ dB. Reference observation was generated by linearly convolving the nonlinearly mapped input signal $f[x]$ with the system $w_\tau$, and the signal-to-noise ratio was set to $\text{SNR} = 60$ dB. The multichannel algorithm with $R = 64$ and $M = 256$ was executed for nonlinear expansions with model order $p = 5$.

We analyzed the performance with respect to the relative error signal attenuation $\text{ESA} = \sigma_{e\tau}^2/\sigma_{w\tau}^2$ measure as well as with respect to the learning of the underlying nonlinear function. As depicted in Fig. 4, the multichannel configuration relying on polynomial modeling of the non-linearity not only suffers from slower and lower convergence but it also undergoes misconvergence in the long...
run. Therefore, we augmented the polynomial modeling by applying Gram-Schmidt orthogonalization to the input signals at hand, i.e., $X^{H}_H = [X_1, X_2, \ldots, X_p]$. As expected, orthogonalization greatly improves the performance and thus verifies the earlier notion regarding the adverse effects of inter-channel correlation. Then we considered the multichannel algorithm relying on the truncated Fourier series. It can be seen that modeling by means of Fourier series enables the algorithm to achieve better performance and convergence rate similar to polynomial modeling with decorrelation. The inherent orthogonality property of the Fourier expansion, however, renders any additional orthogonalization unnecessary.

With the knowledge of the system $W_\tau$ under laboratory conditions, we were able to extract least-squares optimal polynomial and Fourier coefficients, i.e., $a_i = (W^H_\tau W_{\tau,i})/(W^H_\tau W_\tau)$ [9], where $\hat{W}_{\tau,i}$ is the estimate of the $i$-th channel. For a fair comparison, the extracted coefficients were used to reconstruct the inferred underlying nonlinear mappings at block-time $\tau = 6000$, i.e., when all schemes are stable and have converged. It is worth mentioning that Gram-Schmidt orthogonalization when applied to the polynomial model, results in the estimation of an equivalent orthogonal system. In order to extract the nonlinear coefficients it is essential to recover the original filter coefficients. We do so by carrying out the necessary bias correction on all channels of order $i < p$ [1].

We observe in Fig. 5 that the inference of the nonlinear function is consistent with the performance of a given configuration with respect to the ESA measure. Here we also see that the multichannel algorithm was able to best infer the underlying nonlinearity when operated on the basis of the Fourier expansion. The plain polynomial model shows more deviation, however slight, in the linear range from the ground truth. The polynomial model augmented with Gram-Schmidt orthogonalization matches the Fourier model in the linear range but deviates from the ground truth on the fringes of the clipped region. This deviation seems harmless from the ESA perspective for the case at hand, as $|x_i| > 0.4$ represents only a minor percentage of the input signal energy. Nevertheless, such a deviation will be undesirable in the presence of outliers.

5. CONCLUSIONS

In this contribution we addressed the system identification problem when the input signal undergoes an unknown memoryless transformation before convolving linearly with an unknown system. We have, along side the traditional approach of considering power series expansion of the nonlinearity, elucidated the effectiveness of considering a truncated Fourier series. For system identification, we absorb the expansion coefficients into the unknown system and formulate a basis-generic multichannel algorithm which conveniently accommodates both types of modeling. Both of the modeling strategies were analyzed and compared by means of simulation results with respect to relative error signal attenuation and the inference of the nonlinear mapping. We have shown that the multichannel algorithm, when operated on the basis of the Fourier expansion, does not suffer from inter-channel correlation due to the inherent orthogonality property of the expansion basis and thus yields faster convergence. Such is not the case with polynomial modeling that requires an additional orthogonalization stage. The graphical depiction of the inferred nonlinearity also consistently points towards the superiority of the proposed Fourier modeling of the nonlinear transformation.

6. REFERENCES