QUASI PERFECT RECONSTRUCTION FREQUENCY WARPING OPERATOR

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ABSTRACT

In this work we introduce a new frequency warping operator for non-smooth warping function allowing perfect reconstruction. The transformation is based on a previously introduced aliasing compensated frequency warping operator having a residual error because of finite output dimension. By adding some redundancy, the effect of an infinite output dimension can be taken into account in a compressed way, based on an analytical factorization. In the reconstruction process, the additional redundant samples are expanded, making the inverse transform differ from the direct one, but guaranteeing perfect reconstruction.

Index Terms— Frequency Warping, Biorthogonal Transform, Perfect Reconstruction.

1. INTRODUCTION

The flexible tiling of the time-frequency (TF) plane can be addressed by many strategies. When covariance with group delay shift is required, frequency warping is the tool which gives a complete matching between the desired tiling and the obtained analysis [1, 2]. Moreover, warping transforms, thanks to the ideal unitary property, have been used in combination with traditional TF transformations, like filter banks or wavelet transform, in order to generalize their properties [3]. First, an adapted deformation is preliminarily applied to the input signal (warping) and then the TF transformation is performed.

The modeling and computation of warping operators considered as finite length transforms, with major reference to frequency warping, has not been exhaustively treated in literature. The common approach is to approximate them by the decomposition in a nonuniform Fourier transform [4] and an inverse Fourier transform. Nevertheless, many issues about the design of the warping function and reconstruction have not been explored in depth. Unitariness is not satisfied and perfect reconstruction can be achieved by means of iterative methods only.

Recently in [5,6] we illustrated an accurate computational method in case of arbitrary non-smooth warping maps and traced some design criteria. By compensating the aliasing effect involved with nonuniform Fourier transform, frequency warping operator can be computed with a much larger amount of precision, so that unitary property is close to be satisfied. More precisely, the upper bound in reconstruction accuracy achievable by means of the transpose operator is reached, so further improvements are not possible.

Here we propose a new approach. First, the warping operator is conveniently modeled as a non-finite length transform, guaranteeing perfect reconstruction. By exploiting a factorization, the non-finite output representation is compressed in a finite length vector, which is decompressed in the inverse transformation. Perfect reconstruction is ideally achieved except for negligible approximation errors, but direct and inverse operators differ because of the compression operation. So, the transformation is biorthogonal, but could be brought back to be orthogonal one through an expansion.

The next section deals with basic concepts about frequency warping and reviews the approach in [6]. In section 3 we introduce the new operator and discuss some computational aspects. Finally in section 4 some experimental results are illustrated.

2. FREQUENCY WARPING OPERATORS

The discussion is limited to time-discrete signals, the frequency axis is periodic with normalized period of length equal to 1. Frequency warping consists in imposing a deformation of the input signal spectrum according to an arbitrary warping map matching some constraints. The map is defined in the fundamental period [0,1) and then extended to the rest of the frequency axis by imposing that the frequency deviation \( \Delta(f) = w(f) - f \) is a periodic function, that is \( w(f + k) - (f + k) = w(f) - f, \) with \( k \in \mathbb{Z} \), so that \( w(f + k) = k + w(f), \) \( k \in \mathbb{Z} \). In addition the resulting transformation is imposed to be unitary through an orthogonalizing factor.

Starting from the Fourier Transform (FT) kernel for discrete-time spaces \( \mathcal{F}(f, n) = e^{-j2\pi nf}, n \in \mathbb{Z}, f \in [0, 1) \), we introduce the warped Fourier operator kernel:

\[
\mathcal{F}_w(f, n) = \sqrt{w(f)}e^{-j2\pi nw(f)} \quad n \in \mathbb{Z}, f \in [0, 1)
\]
being $\hat{w}$ the derivative of $w$. The inverse operator is equal to the adjoint one, thanks to unitary property. With these definitions, frequency warping can be compactly expressed by through the adjoint Fourier Transform $F^\dagger$:

$$W = F^\dagger F_w$$

that is an orthonormal matrix of infinite dimensions whose entries are:

$$W(m, n) = \int_0^1 \sqrt{\hat{w}(f)} e^{j2\pi mf - nw(f)} df \quad m, n \in \mathbb{Z}.$$  \hfill (1)

However, for a generic sequence $s(n)$, the computation of $F_w s$ would involve infinite terms. Then, we redefine the frequency warping operator by considering as input the space of $N$-tuples of real numbers in $\ell^2(\mathbb{Z}_N)$, while the codomain is $\ell^2(\mathbb{Z})$:

$$W(m, n) = \begin{cases} W(m, n) & n \in \mathbb{Z}_N \\ 0 & n \notin \mathbb{Z}_N \end{cases} \quad (m, n) \in \mathbb{Z}.$$  \hfill (2)

For computational purposes we consider a symmetrical input requirement of considering infinite length output samples. The frequency warping operator defined in (1), being even. The frequency warping operator defined in (1), being invertible by the application of the adjoint transformation is invertible by the application of the adjoint operator $W^\dagger$.

For finite-length input signals, the warped FT kernel is:

$$F_w(f, n) = \sqrt{\hat{w}(f)} e^{j2\pi nw(f)} \quad n \in \mathbb{Z}_N, f \in [0, 1).$$

Since $F^\dagger F_w = I$, then (1) is now defined as:

$$W = F^\dagger F_w$$

and the reconstruction of identity is still satisfied since $W^\dagger W = F^\dagger F^\dagger F_w = I$. The warping operator $W$ can be conveniently represented as:

$$W(m, n) = \begin{cases} T(m, n) & m \in \mathbb{Z}_M \\ E(m, n) & m \notin \mathbb{Z}_M \end{cases}$$

where $T$ is a $M \times N$ truncated version of (2), while $E$ is a $\infty \times N$ error matrix containing the decaying tails of $W(m, n)$ for $m \to \pm \infty$. The introduction of $T$ is aimed to relax the requirement of considering infinite length output samples. The set $\mathbb{Z}_M$ of $M$ consecutive integers must be chosen properly, so that $T$ can still be inverted. Basically, $\mathbb{Z}_M$ should be large enough to include all the significative entries of $W$. In order to preserve most of the input signal energy, we adopt the following standard choice:

$$M > 2 \lceil N/2 \max \hat{w} \rceil$$

then we assume $\mathbb{Z}_M = \{-M/2, \ldots, M/2 - 1\}$ as for the input domain.

For a feasible implementation of $T$, a sampling operation has to be performed on the frequency axis, so that $M$ discrete frequencies $f_k = k/M, k = 0, 1, \ldots, M - 1$ are considered. By introducing the discrete FT $F(k, m) = e^{-j2\pi mk/M}, k, m \in \mathbb{Z}_M$, of size $M \times M$ and the nonuniform discrete FT of size $M \times N$ scaled along rows according to the orthogonalizing factor:

$$F_w(k, n) = \sqrt{\hat{w}(k/M)} e^{j2\pi nw(k/M)} \quad k \in \mathbb{Z}_M, n \in \mathbb{Z}_N$$

the discrete warping operator is represented by:

$$W = F^{-1} F_w = M^{-1} F^\dagger F_w,$$

whose elements, $m \in \mathbb{Z}_M$ and $n \in \mathbb{Z}_N$, are:

$$W(m, n) = \frac{1}{M} \sum_{k=0}^{M-1} \sqrt{\hat{w}(k/M)} e^{j2\pi mk/M - nw(k/M)}, \quad (3)$$

The introduced operator kernels are summarized in Table 1.

Table 1: Summary of operator kernels involved in discrete-time frequency warping for Continuous Frequency (CF) and Discrete Frequency (DF) axis.

<table>
<thead>
<tr>
<th>Fourier Kernel</th>
<th>Warped Fourier Kernel</th>
<th>Warping Kernel</th>
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<tbody>
<tr>
<td>CF $F(f, n) = e^{-j2\pi wf}$</td>
<td>$F_w(f, n) = \sqrt{\hat{w}(f)} e^{-j2\pi nw(f)}$</td>
<td>$W(m, n) = \int_0^1 \sqrt{\hat{w}(f)} e^{-j2\pi mf - nw(f)} df$</td>
</tr>
<tr>
<td>DF $F(k, n) = e^{-j2\pi nk/N}$</td>
<td>$F_w(k, n) = \sqrt{\hat{w}(k/M)} e^{-j2\pi nw(k/M)}$</td>
<td>$W(m, n) = \frac{1}{M} \sum_{k=0}^{M-1} \sqrt{\hat{w}(k/M)} e^{j2\pi mk/M - nw(k/M)}$</td>
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The sampling of the frequency axis in (3) causes periodic repetition in the time domain, which produces aliasing:

$$A(m, n) = \sum_{k \in \mathbb{Z}, k \neq 0} W(m - kM, n) = \sum_{k \in \mathbb{Z}} E(m - kM, n)$$

so that $T = W - A$.

In case of piecewise warping maps containing singularities, which are mostly useful in practical applications, aliasing considerably increases the reconstruction error $\varepsilon_w$. Therefore, aliasing compensation is desirable. In those cases, aliasing related to each singularity is analytical computable through the following factorization [6]:

$$A = PUSVQ$$

where $P$ and $Q$ are diagonal matrix built from vectors $e^{j2\pi mk}$, $m \in \mathbb{Z}_M$ and $e^{-j2\pi nw(\xi)}, n \in \mathbb{Z}_N$ respectively, $V(k, n) = (n/(N/2))^k, n \in \mathbb{Z}_N, k = 0, 1, \ldots, S$ analytically depends
on the warping map and $U$ is a $M$-rows matrix defined by periodically summat-ing the following matrix rows:

$$Y(m,i) = \frac{m - (i+1)}{(M/2)-(i+1)} \quad m \notin Z_M, i = 0,1,\ldots$$

Operator $W$ and $A$ are fast computable, so that $T$ can be conveniently employed. The corresponding error $\varepsilon_T$ is much smaller than $\varepsilon_W$. Operators, their dimensions and their corresponding reconstruction errors are listed in Table 2.

<table>
<thead>
<tr>
<th>Operator</th>
<th>Input</th>
<th>Output</th>
<th>Error</th>
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<tbody>
<tr>
<td>$W$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$| W^TW - I |_2 = 0$</td>
</tr>
<tr>
<td>$W$</td>
<td>$N$</td>
<td>$\infty$</td>
<td>$| W^TW - I |_2 = 0$</td>
</tr>
<tr>
<td>$T$</td>
<td>$N$</td>
<td>$M$</td>
<td>$| T^TT - I |_2 = \varepsilon_T$</td>
</tr>
<tr>
<td>$W$</td>
<td>$N$</td>
<td>$M$</td>
<td>$| W^TW - I |_2 = \varepsilon_W$</td>
</tr>
</tbody>
</table>

where $0$ is the null matrix of suitable dimension. The considered operators allows to nullify reconstruction error if the following is satisfied:

$$H^\dagger RH = E^\dagger E$$

since $T^\dagger T = T^\dagger T + H^\dagger RH$ and $T^\dagger T + E^\dagger E = I$.

Matrix elements $R_{ik} = H^\dagger H_k$ of the combinational matrix $R$ are evidently obtained by considering the products $Y^TP_iP_k Y$ and $Y^TP_iP_k Y$, where $P_i$ refers to $E^{(i)}$. For convenience, matrix $R$ is split in 4 additive components. First the main diagonal is separated from other elements, then both of them are decomposed in terms containing $P_iP_k$ or $P_iP_k$ respectively. So, we define:

$$R^d = Y^TY$$
$$R^{ds} = Y^TP_iP_k Y$$
$$R^{cs} = Y^TP_iP_k Y$$

where $d$ stays for diagonal, while $c$ stays for cross-terms. All these matrix can be easily computed by doing suitable periodic summations on matrix $U$, which was already precomputed for the aliasing compensation. This decomposition has been done because $R^d$ norm is much larger than other norms, thus mainly contributing to $E^\dagger E$. We get:

$$E^\dagger E = G^d + G^{ds} + G^c + G^{cs} = G$$

where:

$$G^d = \sum_{i=1}^{L} \mu_i \Re[H_i R^d H_i]$$
$$G^{ds} = \sum_{i=1}^{L} 2 \Re[H_i R^{ds} H_i]$$
$$G^c = \sum_{i,k=1, i \neq k}^{L} \max(\mu_i, \mu_k) \Re[H_i R^{cs} H_k]$$
$$G^{cs} = \sum_{i,k=1, i \neq k}^{L} 2 \Re[H_i R^{cs} H_k]$$

This formulation would be exact in case matrixes $S$ and $Y$ would have infinite dimensions. Since it has to be limited [6], a residual error could be present. Nevertheless, as the dimensions of $S$ increases, the error fastly decreases below machine error. So, we introduce the variable $K$ being the dimension of the kernel $S$. $K$ also represents the redundancy added for each singularity and affects the overhead in computation caused by the employment of the new operator.

The new transformation belongs to the class of biorthogonal transforms. In addition, by a proper truncation we get operator $T$ which is snug frame with frame ratio very close to 1. Moreover, the added redundant matrix $H$ is actually a
compressed representation of \( \mathbf{E} \), such that the expanded representation could be still obtained. As a drawback, the compressed representation is not in the same domain as for \( \mathbf{T} \) output, therefore processing the transform output could involve some troubles. Instead, this transform could be suitable for compression when it is well matched to the input signal and gives a sparse representation.

4. EXPERIMENTAL RESULTS

In order to evaluate performances we will consider two warping maps. The first one is a simple map used for demonstrative purposes only.

\[
w(f) = \frac{1}{4} (2f^3 - 3f^2 + 5f) \quad f \in [0, 1]
\]  

(4)

It has a single singularity in \( f = 0 \), therefore in the computation of both \( \mathbf{A} \) and \( \mathbf{G} \) a single term is involved. In particular \( \mathbf{G} = \mathbf{G}^d = \mathbf{H}_1 \mathbf{R}^d \mathbf{H}_1 \). The error reconstruction related to this map has been plotted in figure 1 in case of no aliasing compensation (\( \varepsilon_W \)), aliasing compensation (\( \varepsilon_T \)) and for the new operator (\( \varepsilon_G^d = \varepsilon_G \)) respect to the dimension of the kernel \( \mathbf{S} \). With additional 40 samples relative to the projection on \( \mathbf{H}_1 \), the error reconstruction is almost completely eliminated.

Instead, the second considered warping map is useful for practical application, as it can serve for an implementation of the hyperbolic class TF representation. It is piecewise defined by an exponential function and a parabolic interconnection in order to satisfy warping maps constraints:

\[
w(f) = \begin{cases} 
    c_0 + c_1 f + c_2 f^2 & f \in [0, x) \\
    2^{-1} a^{1/(2f-1)} & f \in [x, 1/2) 
\end{cases}
\]  

(5)

Error reconstruction is plotted in figure 2: \( \mathbf{G}^d \) is the most significant term in terms of error reduction, followed by \( \mathbf{G}^c \) and finally by \( \mathbf{G}^{d*} \) while \( \mathbf{G}^{c*} \) is not present. In this example, complete error elimination is more expensive as far as redundancy and complexity are concerned.

5. CONCLUSIONS

A new frequency warping operator carrying perfect reconstruction has been introduced by adding redundancy to frequency warping operator of non-smooth warping maps. Reconstruction of identity is obtained by means of direct and inverse transforms, analytically defined through a factorization, forming a biorthogonal couple. Real implementations have been shown to be effective.

6. REFERENCES