CRAMÉR-RAO LOWER BOUND FOR TIME REVERSAL RANGE ESTIMATORS IN N-MULTIPATH SCATTERING ENVIRONMENTS

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ABSTRACT

As an improvement to the conventional radar range estimation based on a single forward propagation of the probing signal, the paper presents a time reversal (TR) based range estimation technique for radar tracking applications and develops the Cramér-Rao lower bounds (CRLB) for the conventional and TR range estimators in an N-multipath scattering environment, where N may assume any value. The CRLBs quantify possible improvement offered by the TR range estimator over the conventional approach. Stronger secondary multipath improves the performance of the TR range estimator as its CRLB decreases with a higher number of multipaths.

Index Terms—Time Reversal, Cramér-Rao Bounds, Range Estimation, Radar Tracking, and Multipath.

1. INTRODUCTION

In active radar, a signal of known waveform probes the channel and its backscatter (typically modeled as a scaled, delayed, and Doppler shifted version of the probing signal) is used to estimate the target’s parameters including the target’s range (measured in terms of the time delay between the probing signal and its backscatter), its radial velocity (evaluated from the Doppler shift introduced in the backscatter), and the direction of arrival of the reflected signal. Tracking targets based solely on radar range measurements [1] has attracted significant attention recently leading to an increased emphasis on accurate range estimation.

The paper presents a time-reversal (TR) based scheme for estimating the range of a passive target. The motivation for this work comes from the need of improving accuracy in radar range estimation systems operational in strong multipath systems. Uncertainty exists primarily because of multipath and clutter making it difficult to accurately determine the true sources of reflections, therefore, resulting in a poor estimate. TR offers an alternate paradigm by its constructive treatment of multipath. In TR [2], the signal reflected from the target and observed at the radar station during the probing stage is energy normalized, time-reversed, and retransmitted back into the medium. The backscatter of the time reversed signal, obtained from this second TR stage, is used for estimating the range of the target. For array detection and imaging applications, it has been shown previously [3–5] that TR adapts well to high clutter environments with rich multipath and is, therefore, a natural choice for radar range estimation problems. This paper makes three contributions. First, the paper applies TR to range estimation problems in active radar systems operating in strong multipath environments. Second, the paper derives the Cramér-Rao lower bounds (CRLB) for the conventional and TR range estimators by applying the Slepian inequality [6] to an N-multipath scattering environment, where N may assume any value. Third, we verify our theoretical analysis through experimental simulations, which quantify the improvement offered by the TR range estimator. Prior results on the CRB’s for range estimation have been limited to a single/direct propagation (1-path) framework. More recent work by Moura et al. [7] is in the domain of wireless communication systems and is based on the Fisher information matrix for a 2-multipath model. The results presented in the paper are developed in the context of radar systems and are generalized to any order N of the multipath. In our experiments, we specialize our CRB results to a practically important signal shape of a linear frequency modulated (chip) pulse sequence used as the probing signal.

The paper is organized as follows. Section 2 reviews the conventional and TR range estimators with their CRLBs derived in Section 3. Section 4 compares the performances of the two range estimators and Section 5 concludes the paper.

2. SYSTEM MODEL

The range R of an unknown stationary target is estimated from the round-trip time delay τ for the probing waveform to travel out to the target and back to the radar. As such, the range estimation problem is equivalent to accurately estimating the time delay between the transmitted signal and its backscatter. Expressed in terms of the root mean square error στ of the time delay, the root mean square error σR of the range (for an unbiased estimator) is given by σR = (c/2στ), where c is the propagation velocity. Throughout this paper, we use the complex envelope notation in our analysis to account for the carrier frequency, so that the baseband signal f(t) takes the form \( R\{f(t)e^{j\omega_0 t}\} \) after modulation, where \( f(t) \) is the complex envelope of the signal \( f(t) \) and \( \omega_0 = 2\pi f_0 \) with \( f_0 \) the carrier frequency. Notation \( R\{\cdot\} \) refers to the real component of the variable included in the parenthesis. The conventional and TR range estimators are based on the following steps:

1. Forward Probing: The channel is probed with the modulated signal \( f(t)e^{j\omega_0 t} \). For the N-multipath model, the complex envelope of the radar received signal is given by

\[
\tilde{y}^{(N)}(t) = \sum_{n=1}^{N} A_n f(t - \tau - \Delta \tau_n) + v(t),
\]

where \( A_n = |A_n|e^{j\psi_n} \) is the attenuation of the n’th path and includes the complex phase \( \psi_n \). Note that the direct reflection from the target is modeled as \( A_1 f(t - \tau) \) such that \( \Delta \tau_1 = 0 \). The measurement noise \( v(t) \) is Gaussian and white with a PSD...
of $N_0$ within the frequencies $[-B/2, B/2]$ of interest, where $B$ is the bandwidth of the signal. In the frequency domain, Eq. (1) is
\[ \hat{Y}^{(N)}(\omega) = \left( A_1 + \sum_{n=2}^{N} A_n e^{-j\omega \Delta \tau_n} \right) e^{-j\omega \tau} F(\omega) + V(\omega), \]
where capital letters $F(\omega)$ and $V(\omega)$ denote, respectively, the Fourier transforms of $f(t)$ and $v(t)$. In the above equation, the $N$-path channel response is denoted by $G(\omega)$.

2. **Conventional Estimation:** The conventional range estimator is based on the matched filter implementation
\[ \hat{r}^{(cv)}(t) = \int_{-\infty}^{+\infty} f(\lambda + t) \hat{y}^{(N)}(\lambda) d\lambda, \tag{2} \]
where $h^{(cv)}(t) = f(-t)$ is the impulse response of the matched filter. In terms of the transfer function, $H^{(cv)}(\omega) = F^*(\omega)$ for the conventional approach, where * denotes conjugation. The estimate for delay $\tau$ is obtained by solving the optimization problem
\[ \tau^{(cv)} = \max_{\tau} \{ |\hat{r}^{(cv)}(\tau)|^2 \}, \tag{3} \]
where $\tau^{(cv)}$ denotes the estimate of the true delay $\tau$.

3. **TR Probing:** In the TR range estimator, the received signal, given by Eq. (1), is time-reversed (equivalent to phase conjugation in the frequency domain), energy normalized, and retransmitted again to probe the channel a second time. The TR probing signal is given by $k\hat{y}^{(N)}(-t)$ or $k\hat{Y}^{(N)*}(\omega)$ in the frequency domain, where $k$ is the energy normalization factor. The complex envelope of the backscatter of the TR probing signal is given by
\[ \hat{X}^{(N)}(\omega) = \sqrt{N} \hat{Y}^{(N)*}(\omega) G(\omega) e^{-j\omega \tau} + W(\omega), \tag{4} \]
where $W(\omega)$ represents accumulated observation noise. Using the inverse Fourier transform, the received signal (minus noise) for the $N$-multipath propagation model in the time domain is
\[ \hat{x}^{(tr)}(t) = k \sum_{n=1}^{N} |A_n|^2 f(t - \tau) \]
\[ + k \sum_{n=1}^{N} \sum_{m=m+1}^{N} A_n^* A_m f(t - \tau + \Delta \tau_m - \Delta \tau_n) \]
\[ \quad + A_n^* A_m f(t - \tau - \Delta \tau_m + \Delta \tau_n). \tag{5} \]
We use Eq. (5) in Section 3 to evaluate the CRLB for the TR range estimator. Note that the inner summation expands only when $(m > n)$.

4. **TR Estimator:** The TR approach is based on correlating the TR backscatter $\hat{x}^{(N)}(t)$ (Step 3) with the TR probing signal $k\hat{y}^{(N)}(-t)$, i.e.,
\[ \hat{r}^{(tr)}(t) = \int_{-\infty}^{+\infty} k\hat{y}^{(N)}(\lambda + t) \hat{X}^{(N)}(\lambda) d\lambda. \tag{6} \]
In the frequency domain, the transfer function $H^{(tr)}(\omega) = k\hat{Y}^{(N)*}(\omega)$. The TR estimation of delay is obtained from
\[ \hat{\tau}^{(tr)} = \max_{\tau} \{ \hat{r}^{(tr)}(\tau) \}. \tag{7} \]
In the following section, we compare the accuracy of our proposed TR-based algorithm (Steps 3 to 4) with the conventional algorithm, which consists of Steps 1 and 2.

3. **CRAMER RAO LOWER BOUNDS**

Theorem 1 derives the CRLB for the conventional range estimator, while Theorem 2 presents the result for the TR approach. Both cases consider an $N$-multipath environment.

**Theorem 1.** The CRLB for the radar time delay measurement for the conventional range estimator using the $N$-multipath model (Eq. (1)) is given by
\[ \text{var} \{ \sigma_\tau^{(cv)} \} \geq \frac{2\pi N_0}{\sum_{m=1}^{N} \sum_{n=1}^{N} |A_n A_m^*|^2 \Delta \tau_m \Delta \tau_n}, \tag{8} \]
where $\Delta \tau_m = \tau_m - \tau_n$ and $N_0 = \eta_0/B$ is the noise per unit bandwidth with $B$ being the bandwidth of the real-valued probing signal $f(t)$.

**Proof.** Slepian [6] showed that the variance of the estimated time delay in radar satisfies the following inequality
\[ \text{var} \{ \sigma_\tau^{(cv)} \} \geq \frac{1}{(1/N_0) \int_{-\infty}^{+\infty} \left| \frac{\partial}{\partial \tau} \sum_{n=1}^{N} A_n f(t - \tau - \Delta \tau_n) \right|^2 dt}, \tag{9} \]
where $I = \int_{-\infty}^{+\infty} \left[ \frac{\partial}{\partial \tau} \sum_{n=1}^{N} A_n f(t - \tau - \Delta \tau_m) \right] dt$. Interchanging the order of summations and integration gives
\[ I = \sum_{n=1}^{N} \sum_{m=1}^{N} A_n^* A_m \times \int_{-\infty}^{+\infty} \left| \frac{\partial}{\partial \tau} f(t - \tau - \Delta \tau_n) \right| \frac{\partial}{\partial \tau} f(t - \tau - \Delta \tau_m) dt. \tag{10} \]
Expressing $f(\tau)$ in terms of its Fourier transform, Integral I for real valued $f(\tau)$ is given by
\[ I = \sum_{n=1}^{N} \sum_{m=1}^{N} \left( \frac{A_n^* A_m}{2\pi} \right) \int \int F(-\omega_1) F(-\omega_2) e^{j(\omega_1 \Delta \tau_n + \omega_2 \Delta \tau_m)} \times \int e^{j(\omega_1 + \omega_2) t} dt d\omega_1 d\omega_2, \tag{11} \]
where the limits of integration are $[-\infty, \infty]$ in all three integrals. Changing the order of integrals and noting that $\int_{-\infty}^{+\infty} \exp(j(\omega_1 \pm \omega_2) t) dt = 2\pi \delta(\omega_1 \pm \omega_2)$, Eq. (11) for real valued signal $f(t)$ reduces to
\[ I = \sum_{n=1}^{N} \sum_{m=1}^{N} \left( \frac{A_n^* A_m}{2\pi} \right) \int_{-\infty}^{+\infty} |F(\omega_1)|^2 e^{j(\Delta \tau_n - \Delta \tau_m) \omega_1} d\omega_1, \tag{12} \]
which simplifies to Theorem 1.
\[
\frac{\text{III}}{k^2} = \int_{-\infty}^{+\infty} \left\{ \sum_{l=1}^{N} |A_l|^2 f(t-\tau) + \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} [A_n A_m^* f(t-\tau + \Delta \tau_m - \Delta \tau_n) + A_m A_n^* f(t-\tau + \Delta \tau_m + \Delta \tau_n)] \right\} dt \\
\times \left\{ \sum_{l=1}^{N} |A_l|^2 f(t-\tau) + \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} [A_n A_m^* f(t-\tau + \Delta \tau_p - \Delta \tau_q) + A_m A_n^* f(t-\tau + \Delta \tau_p + \Delta \tau_q)] \right\} dt
\]

(18)

As a special case, we consider the 2-multipath model with \( N = 2 \), \( \Delta \tau_1 = 0 \), and \( \Delta \tau_2 = \Delta \tau \) in Corollary 1.1.

**Corollary 1.1.** The CRLB for the conventional range estimator for a 2-multipath model is given by

\[
\text{var}\{\sigma_r^{(\text{con})}\} \geq \frac{2\pi N_0}{\left( |A_1|^2 + |A_2|^2 \right)^2 \beta^2_{t,0} + 2 R(A_1 A_2) \beta^2_{\tau,0}},
\]

(13)
a result similar to the one proved in [7] for the 2-multipath model. Next, we derive the CRLB for the TR range estimator.

**Theorem 2.** The CRLB for the radar time delay measurement for the TR range estimator based on the N-multipath model (Eq. (5)) is

\[
\text{var}\{\sigma_r^{(\text{tr})}\} \geq (2\pi N_0) \left\{ \sum_{n=1}^{N} \sum_{p=1}^{n-1} |A_n|^2 |A_p|^2 \beta^2_{t,0,0} \right\}^{-1}
\]

(16)

\[
+4 \sum_{n=1}^{N} \sum_{p=1}^{n-1} \sum_{q=p+1}^{n} |A_n|^2 R(A_n A_q) \beta^2_{t,\tau,\tau_q}
\]

\[
+2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \sum_{p=1}^{n} \sum_{q=p+1}^{N} \text{Re}(A_n A_m A_m A_q) \beta^2_{t,\tau_m,\tau_m - \Delta \tau_q} + \Delta \tau_q
\]

(15)

\[
+2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \sum_{p=1}^{N} \sum_{q=p+1}^{N} \text{Re}(A_n A_m A_m A_q) \beta^2_{t,\tau_m,\tau_m - \Delta \tau_q} \Delta \tau_q - \Delta \tau_p
\]

**Proof.** Using the Slepian inequality, the variance of the estimated time delay for the TR estimator is given by

\[
\text{var}\{\sigma_r^{(\text{tr})}\} \geq \frac{(1/N_0)^2}{\int_{-\infty}^{+\infty} \left| \frac{\partial}{\partial \tau} x^{(\text{tr})}(t) \right|^2 dt}.
\]

Substituting \( x^{(\text{tr})}(t) \) from Eq. (5), Integral \( \text{III}/k^2 \) is expressed as Eq. (18). Multiplying the different terms and after considerable simplification, we get four expressions of the form

\[
T1 \times T3 = \sum_{n=1}^{N} \sum_{m=n+1}^{N} |A_n|^2 |A_m|^2 \beta^2_{t,0,0}
\]

\[
T1 \times T4 = \sum_{n=1}^{N} \sum_{m=n+1}^{N} \sum_{p=1}^{N} \sum_{q=p+1}^{N} 2|A_n|^2 \text{Re}(A_n A_q) \beta^2_{t,\tau,\tau_q}
\]

\[
T2 \times T4 = \sum_{n=1}^{N} \sum_{m=n+1}^{N} \sum_{p=1}^{N} \sum_{q=p+1}^{N} 2|A_n|^2 \text{Re}(A_n A_q) \beta^2_{t,\tau,\tau_q}
\]

4. PERFORMANCE ANALYSIS

In order to plot the CRLBs of the two range estimators, we consider a radar environment based on the multipath model described in Eq. (1). The probing signal is assumed to be a long pulse with linear frequency modulation (LFM), \( f(t) = f(t)e^{j\omega_0 t} \), where the angular frequency \( \omega_0 = 2\pi f_0 \) and the base chirp frequency \( f_0 = 5 \text{ GHz} \). The complex envelope \( f(t) \) is given by

\[
f(t) = \frac{1}{\sqrt{\tau_0}} \text{Rect} \left( \frac{t}{\tau_0} \right) e^{j\mu t^2}.
\]

(17)

The width of the pulse \( \tau_0 \) is set to 50 \( \mu s \) and parameter \( \mu = B/\tau_0 \) with the bandwidth \( B \) set to 100 MHz. In the time domain, the dimension of the vector representing the LFM waveform \( f(t) \) is given by \( 5B\tau_0 = 25 \text{ Ksamples} \). The values for the attenuation coefficients \( A_n \) and delays \( \tau_n \) used to model different scattering (multipath) environments are listed in Tables 1 and 2, respectively. These values are based on running a finite domain time difference (FDFT) simulation on a selected EM domain and observing the attenuation coefficients and delays for scatterers located at randomly distributed locations. Factor \( A_1 \) is the direct path attenuation coefficient, while \( A_n \), for \( n \geq 2 \), corresponds to the higher order coefficients. For a fixed value of \( N \), the noise power is varied to maintain the same signal-to-noise ratio for the conventional and TR range estimators. Fig. 1 compares
Values for the attenuation coefficients (Aₙ)

<table>
<thead>
<tr>
<th>Order</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-multipath</td>
<td>A₁ = 0.9592 ± 12°, A₂ = 0.9383 ± 30°</td>
</tr>
<tr>
<td>5-multipath</td>
<td>A₃ = 0.9145 ± 67°, A₄ = 0.9053 ± 14°, A₅ = 0.7311 ± 37°</td>
</tr>
<tr>
<td>10-multipath</td>
<td>A₆ = 0.6633 ± 20°, A₇ = 0.5812 ± 23°, A₈ = 0.5762 ± 17°, A₉ = 0.4842 ± 39°, A₁₀ = 0.3742 ± 56°</td>
</tr>
</tbody>
</table>

Table 1. Values for the attenuation coefficients used in plotting the CRLB. The 5-multipath presumes attenuation coefficients for the 2- and 5- multipaths. Similarly, the 10-multipath presumes coefficients for the 2- and 5- multipaths.

<table>
<thead>
<tr>
<th>Order</th>
<th>Multipath delays (∆τₙ) in μs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-multipath</td>
<td>∆τ₁ = 0, ∆τ₂ = 0.035</td>
</tr>
<tr>
<td>5-multipath</td>
<td>∆τ₃ = 0.044, ∆τ₄ = 0.062, ∆τ₅ = 0.065</td>
</tr>
<tr>
<td>10-multipath</td>
<td>∆τ₆ = 0.066, ∆τ₇ = 0.085, ∆τ₈ = 0.122, ∆τ₉ = 0.124, ∆τ₁₀ = 0.131</td>
</tr>
</tbody>
</table>

Table 2. Same as Table 1 but listing the values for time delays. As for the attenuation coefficients, the higher order multipath presumes time delays for the low order multipaths.

the CRLBs for the conventional and TR range estimators with the order N of multipath set to 2, 5, and 10. The plots for the CRLBs for the conventional approach are shown with a dotted line, while the plots for the TR approach are shown with a firm line. Fig. 1 illustrates that the CRLBs for the TR range estimator are significantly lower than the corresponding CRLBs for the conventional estimator as the SNR is varied. This is intuitively pleasing indicating the potential of a superior performance with TR. With N set to 10, the difference in the CRLB (or, alternatively the potential gain with TR) can be as high as 20 dB for a 10-multipath propagation model. A second observation that we make is the positive effect of increased multipath on TR. While there is an increase in the CRLBs of the conventional range estimator with the higher number N of multipath (implying a loss in performance for conventional estimators), the CRLBs of the TR range estimator decrease as N is increased. Lower values for the CRLBs resulting with increased multipath indicate that the TR range estimators use multipath to its advantage. Similar observations are made from the CRLBs plotted in Fig. 2 for a randomly selected set of values for the attenuation coefficients and delays subject to the constraints that |Aₙ₊₁| < |Aₙ| and τₙ₊₁ > τₙ.

5. SUMMARY AND FUTURE WORK

In this paper, we introduced a TR based range estimator for target range estimation in radar applications and derived the CRLBs for both the TR and conventional range estimators for an N-multipath environment, where N may assume any arbitrary value. In strong scattering environments, the CRLB expressions show that the TR range estimator has the potential of estimating the target range more accurately than its conventional counterpart. This is a consequence of the intrinsic ability of TR to adapt the probing waveform to the multipath environment. In this work, our approach was based on a single pair of transmitter and receiver antennas. We intend to extend this work to a framework consisting of antenna arrays to estimate other radar parameters of interest such as the Doppler frequency and angle of arrival associated with moving targets using TR techniques.

6. REFERENCES