A SMOOTHED ANALYSIS APPROACH TO $\ell_1$ OPTIMIZATION IN COMPRESSED SENSING

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ABSTRACT

Recently, [5, 9] theoretically analyzed the success of a polynomial $\ell_1$-optimization algorithm in solving an under-determined system of linear equations. In a large dimensional and statistical context [5, 9] proved that if the number of equations (measurements in the compressed sensing terminology) in the system is proportional to the length of the unknown vector then there is a sparsity (number of non-zero elements of the unknown vector) also proportional to the length of the unknown vector such that $\ell_1$-optimization succeeds in solving the system. In this paper, we consider an alternative performance analysis of $\ell_1$-optimization and demonstrate that linear sparsity is recoverable for a large class of almost deterministic measurement matrices.

Index Terms: compressed sensing, $\ell_1$-optimization

1. INTRODUCTION

In last several years the area of compressed sensing has been the subject of extensive research. The breakthrough results of [5] and [9] theoretically demonstrated that in certain applications (e.g. signal processing in sensor networks) classical sampling at Nyquist rate may not be necessary to perfectly recover signals. These results generated enormous amount of research with possible applications ranging from high-dimensional geometry, image reconstruction, single-pixel camera design, decoding of linear codes, channel estimation in wireless communications, to machine learning, data-streaming algorithms, DNA micro-arrays, magnetoencephalography etc. (more on the compressed sensing problems, their importance, and wide spectrum of different applications can be found in excellent references [1, 7, 12, 19–21, 28]).

In this paper we are interested in the mathematical background of certain compressed sensing problems. As is well known, these problems are very easy to pose and very difficult to solve. Namely, they are as simple as the following: we would like to find $x$ such that

$$Ax = y$$

where $A$ is an $m \times n$ ($m < n$) measurement matrix and $y$ is an $m \times 1$ measurement vector. Standard compressed sensing context assumes that $x$ is an $n \times 1$ unknown $k$-sparse vector (under $k$-sparse vector we assume a vector that has at most $k$ nonzero components). The main topic of this paper will be compressed sensing of the so-called ideally sparse signals (more on the so-called approximately sparse signals can be found in e.g. [6, 30]). We will mostly throughout the paper assume no special structure on the sparse signal (more on sparse signals with special structures the interested reader can find in [1, 13, 14, 18, 25–27]). Also, in the rest of the paper we will assume the so-called linear regime, i.e. we will assume that $k = \beta n$ and that the number of the measurements is $m = \alpha n$ where $\alpha$ and $\beta$ are absolute constants independent of $n$.

A very successful approach to solving (1) that recently attracted a great deal of attention is called $\ell_1$-optimization. Basic $\ell_1$-optimization algorithm finds $x$ in (1) by solving the following $\ell_1$-norm minimization problem

$$\min_{x} \|x\|_1 \quad \text{subject to} \quad Ax = y.$$  \hfill (2)

Quite remarkably, in [5] the authors were able to show that if $\alpha$ and $n$ are given, the matrix $A$ is given and satisfies a special property called the restricted isometry property (RIP), then any unknown vector $x$ with no more than $k = \beta n$ (where $\beta$ is an absolute constant dependent on $\alpha$ and explicitly calculated in [5]) non-zero elements can be recovered by solving (2). As expected, this assumes that $y$ was in fact generated by that $x$ and given to us. The case when the available measurements are noisy versions of $y$ is also of interest [5, 29]. Although that case is not of primary interest in the present paper it is worth mentioning that the recent popularity of $\ell_1$-optimization in compressed sensing is significantly due to its robustness with respect to noisy measurements. (Of course, the main reason for its popularity is its ability to solve (1) for a very wide range of matrices $A$.)

Clearly, having the matrix $A$ satisfy the RIP condition is of critical importance for previous claim to hold (more on the importance of the RIP condition can be found in [4]). For several classes of random matrices (e.g., matrices with i.i.d. zero mean Gaussian, Bernoulli, or even general Sub-gaussian components) the RIP condition is satisfied with overwhelming probability [2, 5]. (Under overwhelming probability we in this paper assume a probability that is no more than a number exponentially decaying in $n$ away from 1.) However, the RIP is only a sufficient condition for $\ell_1$-optimization to produce the solution of (1).

Instead of characterizing the $m \times n$ matrix $A$ through the RIP condition, in [9] the author associates certain polytope with the matrix $A$. Namely, [9] considers polytope obtained by projecting the regular $n$-dimensional cross-polytope using the matrix $A$. It turns out that a necessary and sufficient condition for (2) to produce the solution of (1) is that this polytope associated with the matrix $A$ is $k$-neighborly [9]. Using high-dimensional geometry it is further shown in [9], that if the matrix $A$ is a random $m \times n$ orthoprojector matrix then with overwhelming probability polytope obtained projecting the standard $n$-dimensional cross-polytope by $A$ is $k$-neighborly. The precise relation between $m$ and $k$ in order for this to happen is characterized in [9] as well.
2. PERFORMANCE ANALYSIS OF $\ell_1$-OPTIMIZATION

What we described in the previous section is the standard compressed sensing setup. Carefully reading the previous section one can note that both sets of introductory results [5, 9] rely on a probabilistic choice of the measurement matrix $A$. An outstanding question that remains open relates to obtaining results similar to those from [5, 9] in the so-called deterministic context. Namely, the following question is of fundamental importance:

- Can one identify classes of deterministic measurement matrices $A$ for which the results from [5, 9], related to linearity of recoverable sparsity, still hold?

The results in [5, 9] assume different statistics of the measurement matrix $A$ so that maximal recoverable sparsity by $\ell_1$-optimization is linear with respect to the length of the unknown vector $x$. On the other hand when applied to solve (1), the $\ell_1$-optimization usually succeeds independently of particular statistics of the matrix $A$ (more on this universality from a statistical point of view the interested reader can find in [10]). This suggests that it may in fact be true that the linear sparsity is recoverable for a range of measurement matrices that goes way beyond the Gaussian and Bernoulli matrices considered in seminal works [5, 9] or say the Subgaussian ones considered in [15]. The main contribution of this paper will be determining a significantly larger class of measurement matrices for which the sparsity proportional to the length of the unknown vector $x$ can be recovered by $\ell_1$-optimization. Before we proceed further we briefly mention a few recent results that are somewhat related to the above raised question.

For a deterministic class of measurement matrices it was shown in [8] that $\ell_1$-optimization can recover a sparsity $k = O(\sqrt{m \log m})$ (the original result has an extra log term which in the regime of interest in this paper becomes a constant; we recall that in this paper we always assume that $m = \Theta(n)$ where $\Theta$ is an absolute constant independent of $n$). While this result is an impressive feat it unfortunately falls short of the best possible scaling behavior for recoverable sparsity $k = O(m)$ obtained for statistical measurement matrices in [5, 9]. In recent works [3, 17] the authors introduced another two classes of deterministic matrices for which $\ell_1$-optimization can recover sparsity $k = O(n)$. This is clearly the best possible scaling behavior that one can hope for. However the resulting matrices are restricted to have only binary elements (i.e. each component of these matrices is either zero or one) and of course to have certain structure.

While the results that we will present in the following sections will attempt to give a general answer to the above raised question they will conceptually be of a slightly different flavor when compared to the results that we just mentioned. Namely, instead of determining deterministic measurement matrices we will focus on finding measurement matrices that are in certain sense almost deterministic. Our analysis will resemble on the main idea behind the concept of smoothed complexity introduced for complexity analysis of a simplex algorithm in [22]. Before proceeding further we briefly in the following subsection introduce a characterization of the null-space of the measurement matrix $A$ that will play a key role in a smoothed-like performance analysis that we will present immediately afterwards.

2.1. Null-space characterization of the measurement matrix $A$

The following theorem from [24] provides the null-space characterization of the matrix $A$ that guarantees success of $\ell_1$-optimization (2) in recovering $x$ in (1) (see also, [11, 16, 25]).

**Theorem 1** Assume that an $m \times n$ measurement matrix $A$ is given. Let $x$ be a $k$-sparse vector. Further, assume that $y = Ax$ and that $w$ is an $n \times 1$ vector. Let $K$ be any subset of $\{1, 2, \ldots, n\}$ such that $|K| = k$ and let $\bar{K}_i$ denote the $i$-th element of $K$. Further, let $\bar{K} = \{1, 2, \ldots, n\} \setminus K$. Let $I$ be a $2^k \times k$ sign matrix. Each element of the matrix $I$ is either 0 or 1 but there are no two rows that are identical. Let $I_j$ be the $j$-th row of the matrix $I$. Then (2) will produce the solution of (1) if

$$\langle w \in \mathbb{R}^n | Aw = 0 \rangle \text{ and } \forall K, I_j, w_K < \sum_{i=1}^{n-k} |w_{K_i}|. \quad (3)$$

In the following subsection we will show that the condition (3) from the Theorem 1 will hold for a very wide range of matrices that can act as a basis of the null-space of the measurement matrix.

2.2. Smoothed-like analysis of the null-space characterization

In this section we consider a specific class of matrices $A$ and show that for such a class of matrices condition given in (3) holds. From Theorem 1 it easily follows that certain properties of the null-space of $A$ play crucial role in success of $\ell_1$-optimization. Hence, we start by defining matrix $Z$ as a basis of the null-space of $A$. Clearly, $Z$ is an $n \times (n - m)$ matrix. Typically in statistical context, one assumes that the elements of $A$ are randomly chosen according to certain distribution and therefore the null-space of $A$ and its basis $Z$ can be viewed as random as well. In this paper we also assume that $Z$ is random, however we assume a fairly low degree of randomness. Namely, we will assume that

$$Z = D + E \quad (4)$$

where $D$ is a deterministic matrix and $E$ is a random matrix. We will further assume that the entries of $E$ are i.i.d. Gaussian random variables with zero-mean and variance $\epsilon^2$.

Let $Z_i$ be the $i$-th row of $Z$ and $Z_{ij}$ be the $i, j$-element of $Z$. In the same fashion let $D_i$ be the $i$-th row of $D$ and $D_{ij}$ be the $i, j$-element of $D$. Clearly, $Z_{ij}$ is Gaussian random variable with mean $D_{ij}$ and variance $\epsilon^2$ and all elements of the matrix $Z$ are independent. Since $Z$ is a basis of the null-space of the matrix $A$ it holds $AZ = 0$. Furthermore, any $n \times 1$ vector $w$ from the null-space of $A$ can be represented as $Zv$ where $v \in \mathbb{R}^{n-m}$. Let $I^*_{K}$ denote the event $-\sum_{i=1}^{n-k} 1_{Z_{ik}}, v \leq \sum_{i=1}^{n-k} |Z_{ik}, v|$. Let $S$ be a $2^{n-k} \times (n - k)$ sign matrix. Each element of the matrix $S$ is either 0 or 1 and there are no two rows that are identical. Let $S_{p, i}$ be the $p$-th row of the matrix $S$ and let $S_{p, i}$ be the $p, i$-element of $S$. Further let $C_{K, v}^{(p)} \leq p \leq 2^{n-k}$ denote the polyhedral cones $S_p, v \geq 0, 1 \leq p \leq (n - k)$.

Essentially, for any given constant $\alpha = \frac{\epsilon}{\sqrt{n}}$ we will compute a constant $\beta = \frac{\epsilon}{\sqrt{n}}$ such that

$$\lim_{n \to \infty} P(I^*_{K}, \forall K \subset \{1, 2, \ldots, n\}, |K| = k, \forall v \in \cup_{p=1}^{2^{n-k}} C_{K, v}^{(p)}, s. t. I^*_{K}) = 1 \quad (5)$$

where probability is taken over the elements of $Z$. In order to show that (5) holds for certain values of $\alpha$ and $\beta$ we will actually show that

$$\lim_{n \to \infty} P(I^*_{K}) = 0. \quad (6)$$

where

$$P_I = P(\exists j \in \{1, 2, \ldots, n\}, |K| = k, \exists v \in C_{K, v}^{(p)} \text{ s. t. } I^*_{K})$$

and $I^*_{K}$ denotes the complement of $I^*_{K}$, i.e. it denotes the event $-\sum_{i=1}^{n-k} 1_{Z_{ik}}, v \geq \sum_{i=1}^{n-k} |Z_{ik}, v|$. In what follows we will repeatedly use $P_I$. Our goal is always to show that $\lim_{n \to \infty} P_I = 0$. 3923
Using the union bound over all subsets $K$ and all indexes $j$ we can write

$$P_f \leq \sum_{i=1}^{n} \sum_{j=1}^{2^n} P(\exists v \in \bigcup_{p=1}^{2^{n-k}} C_{K(i)}^p, \text{ s. t. } K(i) \subseteq K)$$

where $K(i)$ is a subset of $\{1, 2, \ldots, n\}$ and $|K(i)| = k$. Clearly the number of these subsets is $\binom{n}{k}$ and hence the first summation on the right side of (7) goes from 1 to $\binom{n}{k}$. Without loss of generality we can assume that the maximum term in the summation on the right hand side of (7) corresponds to the first $k$ rows of the matrix $Z$ and to the row $1$ of matrix $I$ that has all ones. Therefore we can further write

$$P_f \leq \binom{n}{k} \sum_{i=1}^{n} \sum_{j=1}^{2^n} P(\exists v \in \bigcup_{p=1}^{2^{n-k}} C_{i}^p, \text{ s. t. } \sum_{i=k+1}^{n} Z_i v_i \geq \sum_{i=k+1}^{n} |Z_i v_i|)$$

where $C_{i}^p$, $1 \leq p \leq 2^{n-k}$, denote the polyhedral cones $C_{i}^p, \forall k+1 \leq 0, 1 \leq i \leq (n-k)$ (see Figure 1).

![Fig. 1. Cones $C_{i}^p$](image)

Let $E_G$ be the set of all extreme rays of $\bigcup_{p=1}^{2^{n-k}} C_{i}^p$. The function $f(v) = - \sum_{i=1}^{k} Z_i v_i - \sum_{i=k+1}^{n} |Z_i v_i|$ is convex (in fact linear) over the union of cones $\bigcup_{p=1}^{2^{n-k}} C_{i}^p$ and achieves the maximum (up to a scaling constant) on its extreme rays. Hence we have

$$P_f \leq \binom{n}{k} 2^n P(\forall v \in C, \text{ s. t. } f(v) \geq 0) \leq \binom{n}{k} 2^n P(\max_{v \in E} (f(v)) \geq 0).$$

Using the union bound over $v$ we further obtain

$$P_f \leq \binom{n}{k} 2^n P(\max_{v \in E}(f(v)) \geq 0) \leq \binom{n}{k} 2^n \sum_{i=1}^{E_G} P(f(v_i) \geq 0)$$

$$= \binom{n}{k} \sum_{i=1}^{E_G} \sum_{i=k+1}^{n} (Z_i v_i \geq \sum_{i=k+1}^{n} |Z_i v_i|).$$

where $v_i, 1 \leq i \leq |E_G|$ are the extreme rays of $\bigcup_{p=1}^{2^{n-k}} C_{i}^p$. As shown in [23] $E_G \leq \frac{2^{n-k}}{n-m-1}(n-m)$. It is straightforward to show that the probabilities inside the last summation of (8) are insensitive to the norm of $v_i$ [24]. We then finally obtain

$$P_f \leq \binom{n}{k} 2^{n-k} \frac{n-m}{2^{m-1}} \left( \frac{n}{n-k} \right) \sum_{i=k+1}^{n} (Z_i v_i \geq 0, k+1 \leq i \leq m+1).$$

We chose in (9) the cone $Z_i v_i \geq 0, k+1 \leq i \leq m+1$. However the following analysis is insensitive to the choice of the direction of inequalities in the definition of this cone, i.e. it would have the same final result for any other direction of the inequalities. To simplify the notation we set $z_i = Z_i c, \ d_i = D_i c, \ e_i = E_i c, \ 1 \leq i \leq m+1$. Clearly, $z_i$ is a Gaussian random variable with mean $d_i$ and variance $\epsilon^2$. Using the Chernoff bound we then have

$$P_f \leq \binom{n}{k} \sum_{i=k+1}^{n} \frac{1}{2^{m-1}} \sum_{i=k+1}^{n} \sum_{i=m+1}^{n} \left( e_i - \frac{(z_i - d_i)^2}{2 \epsilon^2} \right).$$

where $\mu$ is a constant. Using the entropy approximations of the binomial coefficients $\binom{n}{k} \approx e^{-n H(\beta)}$ and $\binom{n-k}{m} \approx e^{-n H(\beta)(\frac{1}{2})}$ we further have

$$P_f \leq \binom{n}{k} \sum_{i=k+1}^{n} \left( e_i - \frac{(z_i - d_i)^2}{2 \epsilon^2} \right).$$

To show the recoverability of linear sparsity we need to show that when $n \to \infty$ the right hand side in (11) is converging to zero. We first separately analyze $E e_i e^{-\frac{(z_i - d_i)^2}{2 \epsilon^2}}$ and the product terms in (11).

$$E e_i e^{-\frac{(z_i - d_i)^2}{2 \epsilon^2}} = \frac{1}{2^{m-1}} \sum_{i=k+1}^{n} \frac{1}{2 \epsilon^2} \int_{-\infty}^{\infty} e^{-\frac{(z_i - d_i)^2}{2 \epsilon^2}} dz_i = e^{\mu d_i} e^{-\mu^2 \epsilon^2} \int_{-\infty}^{\infty} e^{-\frac{z_i^2}{2 \epsilon^2}} dz_i.$$  

$$= e^{\mu d_i} e^{-\mu^2 \epsilon^2} \left( -\frac{\mu}{\sqrt{\pi}} + \frac{\mu^2}{2 \epsilon^2} \right).$$

Let

$$\xi_k = 2^n e^{-H(\beta)} e^{-\frac{(z_i - d_i)^2}{2 \epsilon^2}} \left( \prod_{i=k+1}^{n} \frac{1}{\sqrt{2 \pi \epsilon}} \right)^{\alpha} \beta \epsilon \int_{-\infty}^{\infty} e^{-\frac{z_i^2}{2 \epsilon^2}} dz_i.$$  

Then we have

$$P_f \leq (\xi_k)^n.$$  

We can now formulate the following theorem.

**Theorem 2 (General condition)** Let $A$ be an $m \times n$ measurement matrix with a basis of its null-space comprised of $Z_{ij}, 1 \leq i \leq n, 1 \leq j \leq (n-m)$ elements. Let $Z_{ij}$ be independent Gaussian random variables with mean $D_{ij}$ and variance $\epsilon^2$. Let $\alpha = \frac{n}{m},$ and $\beta = \frac{k}{n} \leq \frac{1}{2}$ and let $n$ be large. If $D_{ij}, c, \alpha, \beta$ are such that $\xi_k$ is a constant and is strictly less than 1 then any $k$-sparse $x$ which satisfies (1) can be found as the solution of (2). Further, the value of $\beta$ can be obtained as

$$\max \beta$$

s.t. $\xi_k < 1, \mu > 0.$

**Proof** Follows from the previous discussion by combining (12), (13), and (14).

**Corollary 1 (An example for perturbed basis of the null-space)** Let $A$ be an $m \times n$ measurement matrix with a basis of its null-space comprised of $Z_{ij}, 1 \leq i \leq m, 1 \leq j \leq (n-m)$ elements. Let $Z_{ij}$ be independent Gaussian random variables with mean $D_{ij} \geq 1$ and variance $\epsilon^2 < 1$. Further, let $c$ and $D_{ij}$ be constants independent of $n$. Also, let the fraction of nonzeros elements in each column of $D$ be at most $\sqrt{1-k} \delta > 0$. Then for any constant $\alpha = \frac{n}{m}$ there will be an absolute constant $\beta = \frac{k}{n}$ such that $\lim_{n \to \infty} P_f = 0.$
Proof 2 According to Theorem 2 it is enough to show that under the assumptions given in the corollary for any constant \( \alpha < 1 \) there is a constant \( \beta \) such that \( \xi_\beta < 1 \). First, let \( \mu = \frac{\alpha}{\beta} \) which is a huge positive constant. It is then straightforward to see that
\[
\lim_{n \to \infty} e^{\max_{\|x\|_2 \leq 1} \left( \mu \sum_{i=1}^{c} D_i e_i \right)^2} = \lim_{n \to \infty} e^{\max_{\|x\|_2 \leq 1} \left( \mu \sum_{i=1}^{c} D_i e_i \right)^2} = 1. \tag{16}
\]
As we will see below, given (16), it easily follows that showing
\[
\max_{\|x\|_2 \leq 1} e^{-\mu D_i e_i \frac{\|x\|_2^2}{2}} \int_{D_i e_i}^{\infty} e^{-\frac{t^2}{2}} dt \leq \frac{125}{C} \tag{17}
\]
will be enough to show that \( \xi_\beta < 1 \). Let us therefore first focus on proving (17). For any \( k \) with \( 1 \leq i \leq m + 1 \) it is either \(-D_i e_i < \frac{\beta \mu}{\alpha} \leq -\frac{3 \beta \mu}{\alpha} C \) or \(-D_i e_i > \frac{3 \beta \mu}{\alpha} C \) or \(-\frac{3 \beta \mu}{\alpha} C \leq -D_i e_i < -\frac{3 \beta \mu}{\alpha} C \).\n
1. Case \(-D_i e_i < -\frac{3 \beta \mu}{\alpha} C \)
If \(-D_i e_i < -\frac{3 \beta \mu}{\alpha} C \) then \(-D_i e_i + \mu e_i > \frac{3 \beta \mu}{\alpha} C \). Then using well known upper bound for the erf function we have
\[
\left( e^{-\mu D_i e_i \frac{\|x\|_2^2}{2}} \int_{D_i e_i}^{\infty} e^{-\frac{t^2}{2}} dt \right) \leq \left( e^{-\mu D_i e_i \frac{\|x\|_2^2}{2}} \int_{-\frac{3 \beta \mu}{\alpha} C}^{\infty} e^{-\frac{t^2}{2}} dt \right) \leq \frac{125}{C} \tag{18}
\]

2. Case \(-\frac{3 \beta \mu}{\alpha} C \leq -D_i e_i < -\frac{3 \beta \mu}{\alpha} C \)
If \(-\frac{3 \beta \mu}{\alpha} C \leq -D_i e_i < -\frac{3 \beta \mu}{\alpha} C \) then \(-D_i e_i + \mu e_i \leq -\frac{3 \beta \mu}{\alpha} C \), it is either
\[
\left( e^{-\mu D_i e_i \frac{\|x\|_2^2}{2}} \int_{D_i e_i}^{\infty} e^{-\frac{t^2}{2}} dt \right) \leq e^{-\mu D_i e_i \frac{\|x\|_2^2}{2}} \leq e^{-\frac{3 \beta \mu}{\alpha} C} \leq \frac{125}{C} \tag{19}
\]
Combining (18) and (19) we easily have (17). Now if one chooses \( \beta = \frac{1}{\sqrt{2}} \) it clearly follows that \( 2^{\frac{\beta^2}{2}} e^{-H(\beta)e^{-H(\beta)}} \leq \frac{\mu \|x\|_2^2}{\alpha} \) in (14) is a constant that can be upper bounded by a value independent of \( C \). Therefore using (17) and choosing large enough \( C \) one can always make \( \xi_\beta < 1 \). This ends the proof.

The following two key points are worth noting: 1) For the choice of deterministic matrix \( D \) made in Corollary 1, recoverability sparsity does not need to be a function of perturbation \( \epsilon \), i.e. there will be a universal recoverable \( \beta \) no matter how small the constant \( \epsilon \) is; 2) The fraction of nonzeros elements in each column of \( D \) is chosen so that its Frobenius norm, \( \|D\|_F \), is as large as possible. Still, when \( n \to \infty \), \( \|D\|_F \ll \|E\|_F \). It would certainly be of interest to see if the linear sparsity can be recovered if \( \|D\|_F \gg \|E\|_F \).

3. CONCLUSION
In this paper we considered recovery of sparse signals from a reduced number of linear measurements. We provided a theoretical performance analysis of a classical polynomial \( \ell_1 \)-optimization algorithm. Under the assumption that the measurement matrix \( A \) has a deterministic basis of the null-space with components randomly perturbed by a small value, we proved that the sparsity recoverable by \( \ell_1 \)-optimization is proportional to the length of the unknown vector.

4. REFERENCES