OPTIMIZATION AND ITS VARIOUS THRESHOLDS IN COMPRESSED SENSING

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ABSTRACT

Recently, [5, 9] theoretically analyzed the success of a polynomial \(\ell_1\)-optimization algorithm in solving an under-determined system of linear equations. In a large dimensional and statistical context [5, 9] proved that if the number of equations (measurements in the compressed sensing terminology) in the system is proportional to the length of the unknown vector then there is a sparsity (number of non-zero elements of the unknown vector) also proportional to the length of the unknown vector such that \(\ell_1\)-optimization succeeds in solving the system. In this paper, we consider an alternative performance analysis of \(\ell_1\)-optimization and demonstrate that the proportionality constants it provides in certain cases match or improve on the best currently known ones from [8, 9].

Index Terms: compressed sensing, \(\ell_1\)-optimization

1. INTRODUCTION

In last several years the area of compressed sensing has been the subject of extensive research. The breakthrough results of [5] and [9] theoretically demonstrated that in certain applications (e.g. signal processing in sensor networks) classical sampling at Nyquist rate may not be necessary to perfectly recover signals. These results generated enormous amount of research with possible applications ranging from high-dimensional geometry, image reconstruction, single-pixel camera design, decoding of linear codes, channel estimation in wireless communications, to machine learning, data-streaming algorithms, DNA micro-arrays, magneto-encephalography etc. (more on the compressed sensing problems, their importance, and wide spectrum of different applications can be found in excellent references [1, 7, 12, 18–20, 28, 30]).

In this paper we are interested in the mathematical background of certain compressed sensing problems. As is well known, these problems are very easy to pose and very difficult to solve. Namely, they are as simple as the following: we would like to find \(x\) such that

\[
Ax = y
\]

where \(A\) is an \(m \times n\) (\(m < n\)) measurement matrix and \(y\) is an \(m \times 1\) measurement vector. Standard compressed sensing context assumes that \(x\) is an \(n \times 1\) unknown \(k\)-sparse vector (under \(k\)-sparse vector we assume a vector that has at most \(k\) nonzero components). The main topic of this paper will be compressed sensing of the so-called ideally sparse signals (more on the so-called approximately sparse signals can be found in e.g. [6, 31]). We will mostly throughout the paper assume no special structure on the sparse signal (more on the very relevant cases of sparse signals with special structures the interested reader can find in [1, 14, 17, 25–27]). Also, in the rest of the paper we will assume the so-called linear regime, i.e. we will assume that \(k = \beta n\) and that the number of the measurements is \(m = \alpha n\) where \(\alpha\) and \(\beta\) are absolute constants independent of \(n\). A very successful approach to solving (1) that recently attracted a great deal of attention is called \(\ell_1\)-optimization. Basic \(\ell_1\)-optimization algorithm finds \(x\) in (1) by solving the following \(\ell_1\)-norm minimization problem

\[
\min_{x} \|x\|_1
\]

subject to \(Ax = y\) \hspace{1cm} (2)

Quite remarkably, in [5] the authors were able to show that if \(\alpha\) and \(n\) are given, the matrix \(A\) is given and satisfies a special property called the restricted isometry property (RIP), then any unknown vector \(x\) with no more than \(k = \beta n\) (where \(\beta\) is an absolute constant dependent on \(\alpha\) and explicitly calculated in [5]) non-zero elements can be recovered by solving (2). As expected, this assumes that \(y\) was in fact generated by that \(x\) and given to us. The case when the available measurements are noisy versions of \(y\) is also of interest [5, 29]. Although that case is not of primary interest in the present paper it is worth mentioning that the recent popularity of \(\ell_1\)-optimization in compressed sensing is significantly due to its robustness with respect to noisy measurements. (Of course, the main reason for its popularity is its ability to solve (1) for a very wide range of matrices \(A\).)

Clearly, having the matrix \(A\) satisfy the RIP condition is of critical importance for previous claim to hold (more on the importance of the RIP condition can be found in [4]). For several classes of random matrices (e.g., matrices with i.i.d. zero mean Gaussian, Bernoulli, or even general Sub-gaussian components) the RIP condition is satisfied with overwhelming probability [2, 5]. (Under overwhelming probability we in this paper assume a probability that is no more than a number exponentially decaying in \(n\) away from 1.) However, the RIP is only a sufficient condition for \(\ell_1\)-optimization to produce the solution of (1).

Instead of characterizing the \(m \times n\) matrix \(A\) through the RIP condition, in [8, 9] the author associates certain polytope with the matrix \(A\). Namely, [8, 9] consider polytope obtained by projecting the regular \(n\)-dimensional cross-polytope using the matrix \(A\). It turns out that a necessary and sufficient condition for (2) to produce the solution of (1) is that this polytope associated with the matrix \(A\) is \(k\)-neighborly [8, 9]. Using high-dimensional geometry it is further shown in [9], that if the matrix \(A\) is a random \(m \times n\) ortho-projector matrix then with overwhelming probability polytope obtained projecting the standard \(n\)-dimensional cross-polytope by \(A\) is \(k\)-neighborly. The precise relation between \(m\) and \(k\) in order for this to happen is characterized in [8, 9] as well.
It should be noted that one usually considers success of (2) in finding solution of (1) for any given \( x \). It is also of interest to consider success of (2) in finding solution of (1) for almost any given \( x \). To make a distinction between these cases we recall on the following definitions from [9, 10].

Clearly, for any given constant \( \alpha \leq 1 \) there is a maximum allowable value of the constant \( \beta \) such that (2) finds solution of (1) with overwhelming probability for any \( x \). This maximum allowable value of the constant \( \beta \) is called the strong threshold (see [9]). We will denote the value of the strong threshold by \( \beta_s \). Similarly, for any given constant \( \alpha \leq 1 \) one can define the sectional threshold as the maximum allowable value of the constant \( \beta \) such that (2) finds the solution of (1) with overwhelming probability for any \( x \) with a given fixed location of non-zero components (see [9]). In a similar fashion one can then denote the value of the weak threshold by \( \beta_w \). In this paper we determine the values of \( \beta_s, \beta_w \) for the entire range of \( \alpha \), i.e. for \( 0 < \alpha \leq 1 \), for a specific group of randomly generated matrices \( A \).

2. KEY THEOREMS

In this section we introduce two useful theorems that turn out to be of key importance for analysis of the success of \( \ell_1 \)-optimization. First we recall on a null-space characterization of the matrix \( A \) that guarantees that the solutions of (1) and (2) coincide. The following theorem from [23] provides this characterization (similar characterizations can be found in [11, 25, 31, 32]; furthermore, if instead of \( \ell_1 \) one, for example, uses an \( \ell_q \)-optimization (0 < \( q < 1 \)) in (2) then characterizations similar to the ones from [11, 25, 32] can be derived as well [16].

**Theorem 1 (Null-space characterization; General \( x \))** Assume that an \( m \times n \) measurement matrix \( A \) is given. Let \( x \) be a \( k \)-sparse vector whose non-zero components can be both positive or negative. Further, assume that \( y = Ax \) and that \( w \) is an \( n \times 1 \) vector. Let \( K \) be any subset of \( \{1, 2, \ldots, n\} \) such that |\( K \)| = \( k \) and let \( K_i \) denote the \( i \)-th element of \( K \). Further, let \( K = \{1, 2, \ldots, n\} \setminus K \). Let \( x \) be a \( 2^k \times k \) sign matrix. Each element of the matrix \( 1 \) is either 1 or \(-1\) and there are no two rows that are identical. Let \( 1_j \) be the \( j \)-th row of the matrix \( 1 \). Then (2) will produce the solution of (1) if

\[
(\forall w \in \mathbb{R}^n | Aw = 0) \qquad \forall K, j - 1_j K < \sum_{i=0}^{n-k} |w|_{K_i}.
\]

**Remark:** The following simplification of the previous theorem is also well-known. Let \( w \in \mathbb{R}^n \) be such that \( Aw = 0 \). Let \( |w|_{(i)} \) be the \( i \)-th smallest magnitude of the elements of \( w \). Set \( \hat{w} = (|w|_{(1)}, |w|_{(2)}, \ldots, |w|_{(n)})^T \). Then (2) will produce the solution of (1) if

\[
(\forall w \in \mathbb{R}^n | Aw = 0) \quad \forall K, j - 1_j K < \sum_{i=1}^{n-k} |\hat{w}|_i
\]

which according to Theorem 1 (or more precisely according to the remark after Theorem 1) means that the solutions of (1) and (2) coincide with probability 1. For any given value of \( \alpha \in (0, 1] \) a threshold value of \( \beta \) can be then determined as a maximum \( \beta \) such that

\[
\left(\frac{\sqrt{m} - \frac{1}{\sqrt{4\pi\alpha}}}{\sqrt{\sqrt{\sqrt{m}}}}\right)^2
\]

**Remark:** Gordon’s original constant 3.5 was substituted by 2.5 in [21]. Both constants are fine for our subsequent analysis.

3. PROBABILISTIC ANALYSIS OF THE NULL-SPACE CHARACTERIZATION

In this section we probabilistically analyze validity of the null-space characterization given in Theorem 1. In the first subsection of this section we will show how one can obtain the values of the strong threshold \( \beta_s \) for the entire range 0 < \( \alpha \leq 1 \) based on such an analysis. In the second subsection we will determine the values of the weak and the sectional threshold.

3.1. Strong threshold

As masterly noted in [21] Theorem 2 can be used to probabilistically analyze (3). Namely, let \( S \) in (4) be

\[
S_x = \{ w \in S_n^{k-1} | \sum_{i=n-k+1}^{n} \hat{w}_i \geq \sum_{i=1}^{n-k} \hat{w}_i \}
\]

where as earlier the notation \( \hat{w} \) is used to denote the vector obtained by sorting the absolute values of the elements of \( w \) in non-decreasing order. (Here and later in the paper, we assume that \( k \) is chosen such that there is an 0 < \( \alpha \leq 1 \) such that the solutions of (1) and (2) coincide.) Let \( Y \) be an \( (n - m) \) dimensional subspace of \( \mathbb{R}^k \) uniformly distributed in Grassmanian. Furthermore, let \( Y \) be the null-space of \( A \). Then as long as \( w(S_x) < \left( \sqrt{\bar{m}} - \frac{1}{\sqrt{4\pi\alpha}} \right)^2 \left(\frac{\sqrt{\sqrt{\sqrt{m}}}}{\sqrt{\sqrt{\sqrt{m}}}}\right)^2 \), \( Y \) will miss \( S_x \) (i.e. (3) will be satisfied) with probability no smaller than the one given in (5). More precisely, if \( \alpha = \frac{m}{n} \) is a constant (the case of interest in this paper), \( n, m \) are large, and \( w(S_x) \) is smaller than but proportional to \( \sqrt{\bar{m}} \) then

\[
P(\forall w \in R^{dn} | Aw = 0, \sum_{i=n-k+1}^{n} \hat{w}_i < \sum_{i=1}^{n-k} \hat{w}_i) \rightarrow 1
\]

which according to Theorem 1 (or more precisely according to the remark after Theorem 1) means that the solutions of (1) and (2) coincide with probability 1. For any given value of \( \alpha \in (0, 1] \) a threshold value of \( \beta \) can be then determined as a maximum \( \beta \) such that

\[
\left(\frac{\sqrt{m} - \frac{1}{\sqrt{4\pi\alpha}}}{\sqrt{\sqrt{\sqrt{m}}}}\right)^2
\]

This maximum \( \beta \) will be exactly the value of the strong threshold \( \beta_s \). If one is only concerned with finding a possible value for \( \beta_s \) it is easy to note that instead of computing \( w(S_x) \) it is sufficient to find its upper bound. However, as we will soon see, to determine as good values of \( \beta_s \) as possible, the upper bound on \( w(S_x) \) should be as tight as possible. The main contribution of this work and [24] is a fairly precise estimate of \( w(S_x) \).
Let $|\mathbf{h}|(i)$ be the $i$-th smallest magnitude of elements of $\mathbf{h}$. Set $\hat{\mathbf{h}} = (|\mathbf{h}|(1), |\mathbf{h}|(2), \ldots, |\mathbf{h}|(n))^T$. Further, let $\mathbf{z} \in \mathbb{R}^n$ be a column vector such that $z_i = 1, 1 \leq i \leq (n-k)$ and $z_i = -1, n-k+1 \leq i \leq n$. Using the Lagrange duality theory it is shown in [24] that there is a $c_s = (1-\theta_s)n \leq (n-k)$ such that

$$w(S_s) = E \max_{w \in S_s} \sum_{i=1}^n \hat{h}_i|w_i| \leq O(\sqrt{n}e^{-c_n \text{const}}) + \sqrt{E \sum_{i=1}^n \hat{h}_i^2} - \frac{(E(\mathbb{h}^T \mathbf{z}) - E\sum_{i=1}^n \hat{h}_i)^2}{n-c_s} \tag{7}$$

where $\hat{h}_i$ is the $i$-th element of vector $\hat{h}$ and $\text{const}$ is an absolute constant independent of $n$. When $n$ is large the first term on the right hand side of inequality in (7) goes to zero. Finding a good $c_s$ and computing the second term on the right hand side of inequality in (7) is then enough to determine a valid upper bound on $w(S_s)$ and therefore to compute the values of the strong threshold. Such results are established in [24] relying on [3, 22]. The following theorem from [24] then summarizes how an attainable value of the strong threshold can be computed.

**Theorem 3** (Strong threshold) Let $A$ be an $m \times n$ measurement matrix in (1) with the null-space uniformly distributed in the Grassmanian. Let the unknown $\mathbf{x}$ in (1) be $k$-sparse. Let $k, m, n$ be large and let $\alpha = \frac{m}{n}$ and $\beta = \frac{k}{n}$ be constants independent of $m$ and $n$. Let erfinv be the inverse of the standard error function associated with zero-mean unit variance Gaussian random variable. Further, let $\epsilon > 0$ be an arbitrarily small constant and $\hat{\theta}_w$, $\hat{\beta}_w \leq \hat{\theta}_w \leq 1$ be the solution of

$$(1-\epsilon)(1-\beta_w)\sqrt{\frac{\pi}{2}}e^{-\text{erfinv}(1-\theta_w)^2} - \sqrt{2\text{erfinv}}((1+\epsilon)(1-\theta_w)) = 0. \tag{10}$$

If $\alpha$ and $\beta_w$ further satisfy

$$\alpha > \frac{1}{\sqrt{2\pi}} \left( \sqrt{2\pi} + 2\frac{\sqrt{2\text{erfinv}(1-\hat{\theta}_w)^2}}{\text{erfinv}(1-\hat{\theta}_w)} - \sqrt{2\pi(1-\hat{\theta}_w)} \right) \tag{11}$$

then the solutions of (1) and (2) coincide with overwhelming probability.

**Proof 1** Follows from the previous discussion and analysis presented in [24].

The results for the strong threshold obtained from the above theorem as well as the best currently known ones from [8, 9] are presented on Figure 1. As can be seen, the threshold results obtained from the previous analysis match those from [8, 9].

**3.2. Weak and sectional thresholds**

In this subsection we present the weak and sectional equivalents to Theorem 3. We start with the weak threshold theorem.

**Theorem 4** (Weak threshold) Let $A$ be an $m \times n$ measurement matrix in (1) with the null-space uniformly distributed in the Grassmanian. Let the unknown $\mathbf{x}$ in (1) be $k$-sparse. Further, let the location and signs of nonzero elements of $\mathbf{x}$ be arbitrarily chosen but fixed. Let $k, m, n$ be large and let $\alpha = \frac{m}{n}$ and $\beta_w = \frac{k}{n}$ be constants independent of $m$ and $n$. Let erfinv be the inverse of the standard error function associated with zero-mean unit variance Gaussian random variable. Further, let $\epsilon > 0$ be an arbitrarily small constant and $\hat{\theta}_w$, $\hat{\beta}_w \leq \hat{\theta}_w \leq 1$ be the solution of

$$(1-\epsilon)(1-\beta_w)\sqrt{\frac{\pi}{2}}e^{-\text{erfinv}(1-\theta_w)^2} - \sqrt{2\text{erfinv}}((1+\epsilon)(1-\theta_w)) = 0. \tag{10}$$

If $\alpha$ and $\beta_w$ further satisfy

$$\alpha > \frac{1}{\sqrt{2\pi}} \left( \sqrt{2\pi} + 2\frac{\sqrt{2\text{erfinv}(1-\hat{\theta}_w)^2}}{\text{erfinv}(1-\hat{\theta}_w)} - \sqrt{2\pi(1-\hat{\theta}_w)} \right) \tag{11}$$

then the solutions of (1) and (2) coincide with overwhelming probability.

**Proof 2** Follows from the analysis presented in [24].

The results for the weak threshold obtained from the above theorem as well as the best currently known ones from [8, 9] are presented on Figure 2. As can be seen, the threshold results obtained from the previous analysis match those from [8, 9].
standard error function associated with zero-mean unit variance Gaussian random variable. Further, let $\epsilon > 0$ be an arbitrarily small constant and $\frac{1}{2} \pi \theta_{\sec} \leq \frac{1}{2} \beta_{\sec} \leq 1$ be the solution of
\begin{equation}
(1 - \epsilon)(1 - \beta_{\sec}) \sqrt{\frac{2}{\pi} e^{-\left(\frac{\theta_{\sec}}{\beta_{\sec}}\right)^2}} - \sqrt{\frac{2}{\pi} e^{-\left(\frac{\theta_{\sec}}{1 - \beta_{\sec}}\right)^2}} - \sqrt{2} \text{erf}(1 + \epsilon) \left(1 - \frac{1 - \theta_{\sec}}{1 - \beta_{\sec}}\right) = 0.
\end{equation}
If $\alpha$ and $\beta_{\sec}$ further satisfy
\begin{equation}
\alpha > \frac{1 - \beta_{\sec}}{2\pi} \left(\sqrt{2\pi} + 2\sqrt{\frac{2}{\pi} e^{-\left(\frac{\theta_{\sec}}{1 - \beta_{\sec}}\right)^2}} - \sqrt{2\pi} \frac{1 - \theta_{\sec}}{1 - \beta_{\sec}}\right)
+ \beta_{\sec} - \left(1 - \beta_{\sec}\right) \sqrt{\frac{2}{\pi} e^{-\left(\frac{\theta_{\sec}}{\beta_{\sec}}\right)^2}} - \sqrt{2\pi} \frac{\beta_{\sec}}{\beta_{\sec}}\right)^2 \frac{1}{\theta_{\sec}}
\end{equation}
then the solutions of (1) and (2) coincide with overwhelming probability.

**Proof 3** Follows from the analysis presented in [24].

The results for the sectional threshold obtained from the above theorem as well as the best currently known ones from [8, 9] are presented on Figure 3. As can be seen, the threshold results obtained from the previous analysis slightly improve on those from [8, 9].

![Fig. 3. Sectional threshold, $l_1$-optimization](image)

### 4. CONCLUSION

In this paper we considered recovery of sparse signals from a reduced number of linear measurements. We provided a theoretical performance analysis of a classical polynomial $l_1$-optimization algorithm. Under the assumption that the measurement matrix $A$ has a basis of the null-space distributed uniformly in the Grassmannian, we derived lower bounds on the values of the recoverable strong, weak, and sectional thresholds in the so-called linear regime, i.e. in the regime when the recoverable sparsity is proportional to the length of the unknown vector. Obtained threshold results are comparable to the best currently known ones.

### 5. REFERENCES


