ON THE OUTAGE AND DIVERSITY-MULTIPLEXING TRADEOFF OF BROADCAST CHANNELS WITH 1-BIT FEEDBACK

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ABSTRACT
In this paper, a single antenna downlink channel with \( K \) users is analyzed in the presence of Rayleigh flat fading. A limited channel state information (CSI) feedback scheme is considered, where only 1-bit feedback per user is available at the base station for each fading block. We study the outage performance of such channels. The two cases of short-term and long-term average power constraints are analyzed. Closed-form expressions for the outage probability of the 1-bit feedback scheme are presented. At high SNR, the diversity-multiplexing tradeoff (DMT) is explicitly derived. We show that for the short-term average power constraint case, with only 1-bit feedback, the scheme achieves not only a DMT that increases linearly with the number of users in the system, but also the same tradeoff as if perfect CSI feedback were available at the base station. For long-term average power constraint, it is concluded that power allocation based on the feedback bit is crucial in further improving the DMT. An additional factor of two in the DMT is achieved compared with the short-term average power constraint case.

Index Terms—Outage probability, limited feedback, multiuser diversity, power allocation, diversity-multiplexing tradeoff.

1. INTRODUCTION

In wireless communications, ergodic and outage capacities are key information-theoretical notions in fading channels [1]. Ergodic capacity typically applies to non-real-time data services and measures the maximum achievable rate on a time-varying fading channel (obtained by encoding over multiple fading realizations). This concept is thus not suitable for applications with stringent delay constraints, such as real-time voice or video services. These issues motivate the concept of outage capacity, defined as the largest rate for which the probability of outage (i.e., the probability that the transmission rate is larger than the instantaneous rate supported by the channel) is under a specified tolerable value [1]. Therefore, compared with ergodic capacity, which refers to long-term average rates, outage capacity emphasizes “instantaneous” rates.

When considering multiuser scenarios over fading channels, e.g., cellular networks, where the channel between a base station and each user experiences independent time variations due to fading, a further key idea is multiuser diversity. Multiuser diversity captures channel fluctuations in order to enhance the throughput of downlink or uplink transmissions. In fact, it has been proved that in cellular systems, serving the user with the best instantaneous channel is optimal in terms of the ergodic sum-rate [2] [3]. But this requires full CSI knowledge for all users at the transmitter, an inherent problem for the optimal multiuser diversity based scheduling.

Various schemes have been proposed in order to reduce the feedback overhead while, at the same time, seeking to preserve capacity (see review in [4]). Among them, a common approach prescribes feedback of a quantized version of the CSI [5]. Recently, a scheme with feedback limited to 1-bit was proposed in [6], analyzed in [7], and further studied by considering the effect of feedback delay in [8]. These works prove that the 1-bit feedback scheme suffers from a negligible loss in terms of multiuser diversity gain as compared to the full CSI feedback scheme, namely, it preserves the optimal scaling law of \( \log \log K \) with the number of users \( K \) [6–8].

Previous work on the downlink channel with 1-bit feedback has focused, as described above, on ergodic channels. In this paper, we focus instead on the study of the outage of such channels and investigate the diversity-multiplexing tradeoff (DMT) [9] of the 1-bit feedback scheme. We consider two different power constraints: short-term average power constraint and long-term average power constraint. For short-term average power constraint, we derive an explicit expression for the outage probability, and then evaluate their asymptotic behavior by finding the DMT. It is shown that even with only 1-bit feedback per user per fading block, and short-term average power constraint, the 1-bit feedback scheme achieves a linear DMT with the number of users in the system for the broadcast multiuser scenario. More importantly, it achieves the same DMT as that of a system with full CSI feedback at the base station and short-term average power constraint [10]. Inspired by [11, 12], we then consider power allocation according to different CSI feedback from users under a long-term average power constraint. We extend the results and show that this strategy doubles the DMT that achieved for the short-term average power constraint case.

2. SYSTEM MODEL

We consider a discrete-time fading broadcast channel where a single-antenna base station communicates with \( K \) single-antenna users. All users’ channels are assumed to be homogeneous and experience independent block Rayleigh flat fading. Accordingly, the fading processes are independent among different users, and the block duration is sufficiently small so as to guarantee that the fading gains remain constant during one block and vary from block to block. The signal received by user \( k \) at a given time \( t \) is described as

\[
y_k = h_k x + n_k, \quad k = 1, \ldots, K
\]

where \( h_k \sim \mathcal{CN}(0,1) \) is the stochastic process representing the fading experienced by user \( k \) within the given fading block, \( n_k \sim \mathcal{CN}(0,1) \) is the stochastic process representing the fading experienced by user \( k \) within the given fading block.
$CN(0,1)$ is complex additive white Gaussian noise with unit variance and assumed statistically independent among different users and over time.

We assume that each user is aware of its own fading power level $|h_k|^2$ based on perfect channel estimations, and compares it to a prescribed threshold $\alpha$. If the fading power $|h_k|^2$ is larger than the threshold $\alpha$, the user feeds back a single bit of “1” through a delayless reliable uplink channel. Otherwise, it feeds back a single bit of “0”. The base station receives all the feedback bits and randomly chooses one of the users with feedback bit “1” for transmission. In case there are no users with channel gains above the threshold $\alpha$, then one user from among all the users is chosen for transmission. Both short-term average power constraint and long-term average power constraint are considered. For short-term average power constraint, during each of the fading block, the transmitted signal $x$ is a zero mean complex Gaussian random variable with power $\frac{1}{P} \sum_{k=1}^{n} |x_k(H_k)|^2 \leq P, \forall H$ (average wrt message set); for long-term average power constraint, different power is adaptively allocated to the chosen user during each of the fading block, according to the CSI feedback bits from all users. When there is at least a user with CSI above the threshold, the chosen user for transmission uses a transmission power of $P_1$; when there is no user above the threshold, the chosen user uses a transmission power of $P_2$. The average transmitting power over fading blocks is $E_{h_k} [ \frac{1}{P} \sum_{k=1}^{n} |x_k(H_k)|^2 ] \leq P$ (average wrt message set and channels). It is noted that this scheduling strategy differs from [6–8] in the sense that, these references, considering ergodic sum-rate, assume for simplicity that no user is served in the absence of useful (“1”) feedback, and proved that this choice does not affect the sum-rate scaling law for both instantaneous and outdated 1-bit feedback cases. Also, power allocation due to different feedbacks from users does not have an impact on the sum-rate scaling law in the references, while it has an important role in terms of DMT in our work for the case of long-term average power constraint.

3. SHORT-TERM AVERAGE POWER CONSTRAINT

In this section, we derive the outage probability for the broadcast channels with 1-bit feedback and short-term average power constraint. Moreover, we analyze the DMT of the scheme.

3.1. Outage Probability

Consistent with the scheduling mechanism in Sec II, a user is chosen for transmission from among those with channel gains exceeding a prescribed threshold or from among all users, if no channel gain exceeds the threshold. In either case, let the transmission power be $P$ and the rate transmitted by the base station be $R$.

**Proposition 1** The outage probability of the broadcast channels with 1-bit feedback, short-term average power constraint $P$, $K$ users, threshold $\alpha \geq 0$, and a prescribed rate $R$, is given by (2) shown at the bottom of the page.

\[ \epsilon = \epsilon_1 \times \Pr(N > 0) + \epsilon_0 \times \Pr(N = 0), \]  

where

\[ \epsilon_1 = \Pr [ \log (1 + \alpha^2 P) < R | |h|^2 \geq \alpha], \]  

\[ \epsilon_0 = \Pr [ \log (1 + \alpha^2 P) < R | |h|^2 < \alpha], \]

are the probabilities of outage conditioned on the feedback bits. Notice that we have dropped the subscript $k$ due to the statistical equivalence of different users. Since we are considering a Rayleigh fading channel, $|h|^2$ is an exponential distribution random variable, equation (4) becomes

\[ \epsilon_1 = \Pr \left( |h|^2 < \frac{e^R - 1}{P} | |h|^2 \geq \alpha \right) = 1 - \exp \left( \alpha - \frac{e^R - 1}{P} \right), \]

if $R > \log(1 + P \alpha)$; and $\epsilon_1 = 0$, if $R \leq \log(1 + P \alpha)$. Solving (5) in a similar way, we have

\[ \epsilon_0 = \begin{cases} \frac{1 - \exp \left( \frac{-e^R - 1}{P} \right)}{1 - e^{-\alpha}} & \text{if } R \leq \log(1 + P \alpha) \\ 1 & \text{otherwise} \end{cases}. \]

The probability that at least one user has its channel power gain $|h|^2$ above the threshold $\alpha$ is

\[ \Pr(N \geq 1) = 1 - \Pr(N = 0) = 1 - \Pr(|h|^2 < \alpha)^K = 1 - (1 - e^{-\alpha})^K, \]

so that from (3) we easily get (2).

3.2. Diversity-Multiplexing Tradeoff

To gain more insight into the system performance, we resort to asymptotic analysis (high SNR) of the DMT of the 1-bit feedback scheme. The notion of DMT was introduced in [9] in order to characterize the performance of transmission schemes over block fading channels for high SNR. It describes the tradeoff between the multiplexing gain $r = \lim_{P \rightarrow \infty} \frac{\log P}{\log R}$, where $R(P)$ is the rate at power $P$, and the diversity gain $d = -\lim_{P \rightarrow \infty} \frac{\log e(P)}{\log P}$, where $e(P)$ is the outage probability at power $P$.

**Proposition 2** The DMT of the broadcast channels with 1-bit feedback, given $K$ users and short-term average power constraint, is given by

\[ d(r) = K(1 - r)^{+}, \]

where $(x)^+$ equals $x$ for $x \geq 0$ and 0 otherwise.

**Proof:** See Appendix for the proof sketch (Detailed proof is provided in [13]).

**Remark 1:** The DMT is the relation between the sum-rate and the slope of the outage probability curve at high SNR. In [10], reasoning on the dual problem of the multiple access channel with short-term average power constraint, the author shows that in the broadcast channel, when full CSI is available at the base station, the optimal DMT is $d(r) = K(1 - r)^+$. Proposition 2 then states that
even with only 1-bit feedback per user, not only the DMT of the scheme with short-term average power constraint increases linearly with the number of users in the system, but also it achieves the same DMT as that of a system with full CSI feedback at the base station [10].

4. LONG-TERM AVERAGE POWER CONSTRAINT

Different with the short-term average power constraint where the average power of codewords for each of the fading block is $P$, the use of long-term average power constraint in the 1-bit feedback scheme is discussed in this section and impacts in terms of the DMT has been shown.

4.1. Outage Probability

The following proposition holds for the outage probability:

**Proposition 3** The outage probability of the broadcast channels with 1-bit feedback, long-term average power constraint $P$, $K$ users, threshold $\alpha \geq 0$, and a prescribed rate $R$, is given by

$$\epsilon = \epsilon_1 \times \Pr(N > 0) + \epsilon_0 \times \Pr(N = 0),$$

where

$$\epsilon_1 = \Pr[\log (1 + |h|^2 P_1) < R | |h|^2 \geq \alpha]$$

$$= \begin{cases} 0 & \text{if } R \leq \log(1 + P_1 \alpha) \\ 1 - \exp\left(\frac{\alpha - e^{P_1}}{P_1}\right) & \text{otherwise} \end{cases},$$

$$\epsilon_0 = \Pr[\log (1 + |h|^2 P_0) < R | |h|^2 < \alpha]$$

$$= \begin{cases} 1 - \exp\left(\frac{1 - \alpha}{1 - e^{-\alpha}}\right) & \text{if } R \leq \log(1 + P_0 \alpha) \\ 1 & \text{otherwise} \end{cases}$$

are the probabilities of outage conditioned on the feedback bits.

**Proof:** Recall that long-term average power constraint implies that different power of $P_1$ and $P_0$ are used for transmission for the cases when one of the users’ channel gains exceeds the threshold ($N > 0$) and when no channel gain exceeds the threshold ($N = 0$), respectively. The outage probability $\epsilon$ for this case is then (10), where $\epsilon_1$ and $\epsilon_0$ are defined similar to (6) and (7), respectively, and can be derived in a similar way.

4.2. Diversity-Multiplexing Tradeoff

Previous works have shown that for a point-to-point link, CSI feedback can dramatically improves the DMT. In particular, [11, 12] focus on the DMT for point-to-point links with partial CSI at transmitter. As a special case of the result there, for a SISO system with 1-bit feedback to the transmitter, the optimal DMT is $d(r) = 2(1 - r)^{+}$. In order to achieve this result, power allocation based on the feedback is a key factor. In this subsection, we look at the DMT for the broadcast channels with 1-bit feedback per user per fading block, and show the important role of power allocation on the DMT.

**Proposition 4** The DMT of broadcast channels with 1-bit feedback, given $K$ users and long-term average power constraint, is lower bounded by

$$d_p(r) = 2K(1 - r)^{+}.$$  

**Proof:** Inspired by [11, 12], we look at a specific power allocation scheme, where we transmit $P_1 = P/2$ when there is at least one user above the threshold $\alpha$, and $P_0 = P_1/Pr(N = 0) = \frac{P}{2(1 - e^{-\alpha})}$ when no user has a channel gain above the threshold. We also choose the threshold $\alpha$ to guarantee that there is no outage occur during the transmission when there is at least a user above the threshold $\alpha$. For the reason of limited space, details of the proof are shown in [13].

**Remark 2:** Compared to the results in [11, 12], our result shows that in the downlink of a multiuser scenario with $K$ users and long-term average power constraint, the DMT increases linearly with the number of users in the system. The key factor here is that in the multiuser scenario, the probability that all users’ channel gains are below the threshold is drastically reduced compared to point-to-point communication, which brings the improvement of DMT as compared to [11, 12]. It is worth noticing that for the special case of our result when there is only one user in the system ($K = 1$), a DMT of $d_p(r) = 2(1 - r)^{+}$ is achieved, which is the same as in [11, 12]. Besides, compared to the result for the case with short-term power constraint in Sec. III, an additional factor of two in the DMT is achieved. The intuition behind this result is that less power is used for transmissions when the chosen user is with a good channel condition and the saved power is used to lower the outage probability when all users are in bad channel conditions.

5. NUMERICAL RESULTS

The outage probability $\epsilon$ of the broadcast channels with 1-bit feedback is shown in Fig. 1 as a function of SNR $r$. For both short-term and long-term average power constraints, different number of users $K$ and fixed outage rate $R = 3$ bits/sec/Hz. For all cases, we choose the threshold $\alpha$ to guarantee that there is no outage occur when there is at least a user above the threshold (as in the Proof of Proposition 2, 4); For long-term average power constraint case, we choose $P_1 = P/2$ when there is at least one user above the threshold $\alpha$, and $P_0 = P_1/Pr(N = 0) = \frac{P}{2(1 - e^{-\alpha})}$ when no user has a channel gain above the threshold, as in the Proof of Proposition 4. It is seen that the outage probability slopes at the high SNR regime keep increasing with the number of users, for both cases of short-term and long-term average power constraints, indicating an increase of the diversity. Besides, with power allocation according to different CSI feedback from users under the long-term average power constraint, additional diversity gains are achieved compared to the case with the short-term power constraint. Note that in Fig. 1, the short-term average power constraint strategy achieves better outage performance at low SNR regime compared to long-term average power constraint case. This is because with the specific power allocation scheme that we use in Proposition 4, we do not fully utilize the total long-term average transmission power $P$, especially at low SNR regime.

The DMT for different feedback schemes and power constraints in broadcast channels (1-bit feedback with short-term average power constraint (9), no CSI feedback [9], full CSI feedback with short-term power constraint [10] and 1-bit feedback with long-term average power constraint (13)) are shown in Fig. 2. The DMT for point-to-point communication with 1-bit feedback [11, 12] is also shown for reference. As can be seen, with multiuser in the system, the diversity increase linearly with the number of users, and power control is a key factor to further improve the diversity.

6. CONCLUDING REMARKS

In this work, the outage of a SISO broadcast channel with quasi-static Rayleigh fading and CSI feedback limited to only 1-bit is investigated. Closed-form expressions for the outage probabilities of the 1-bit feedback scheme are presented for any number of users, the
6.1. Proof of Proposition 2

According to the scheduling mechanism in Sec II, each user compares its own fading power to a prescribed threshold $\alpha$, and decides its feedback bit. We choose the threshold $\alpha$ to guarantee that there is no outage occur during the transmission when there is at least a user above the threshold $\alpha$, so that $R = \log(1 + P\alpha)$. Thus the threshold is a function of power $P$, and $\alpha = \frac{1}{K} \log(1 + P\epsilon)$. With the chosen threshold $\alpha$, the probabilities of outage conditioned on the feedback bits are $\epsilon_1 = 0$, and $\epsilon_0 = \frac{1 - \exp(-\frac{\epsilon_0 R}{K})}{1 - \exp(-\frac{\epsilon_0}{K})}$. Therefore the outage probability is

$$\epsilon = \epsilon_1 \times \Pr(N > 0) + \epsilon_0 \times \Pr(N = 0) = \left(1 - e^{-\frac{R}{K}}\right)^K. \quad (14)$$

Setting the rate to $R = r \log P$ (see [9]), for multiplexing gain $r < 1$, the diversity $d(r)$ is given by

$$d(r) = -\lim_{P \to \infty} \frac{\log \epsilon(\alpha, P, K, r \log P)}{\log P} = K(1 - r). \quad (15)$$

Alternatively, if the multiplexing gain is $r \geq 1$, plug in $R = r \log P$ into (14), it is easily shown that the diversity is $d = 0$.

7. REFERENCES