QUANTIFYING INFORMATION RATE LOSSES WITH ZERO-FORCING AND MAXIMUM-LIKELIHOOD DETECTORS

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ABSTRACT
MIMO systems have caught a lot of attentions mainly because of the boost of the information rate. However, so far the existing results only consider continuous signals at the receiver side. The effect of detectors on the mutual information has not been addressed. In this paper, we study the mutual information between the transmitted discrete signal and the quantized signal given by maximum-likelihood or zero-forcing detectors over MIMO channels. To differentiate from the mutual information without quantization step, we refer the one here as post-detection mutual information (PMI). The approximations for PMI with MLD or ZFD and the closed form of the PMI with ZFD are derived. We show that the reliable data rates that can be transmitted through the MIMO channels with MLD or ZFD is actually reduced by the presence of quantization. The approximations proposed in the paper match well with the simulation results in the mid to high SNR region.

Index Terms— Post-detection mutual information, QAM constellation, maximum-likelihood detector, zero-forcing detector, MIMO channel

1. INTRODUCTION
By taking advantage of the spatial diversity in wireless communication or data storage systems, multiple inputs and multiple outputs (MIMO) systems offer a significant improvement in the reliable data rates compared with the single input and single output (SISO) system [1, 2]. However even in MIMO channels, different detectors in the receivers induce different performance. One of the important criteria to quantify the performance of different detectors is the practical mutual information (MI) between the transmitted signal and the detected signal, which determines how much data rates can be transmitted reliably through the communication system.

Actually, there is a lot of research work discussing the information rates of MIMO channels. But all of the previous work assumes that the signal at the receiver is continuous, which is less practical for real systems. For example, when both the transmitted and received signals are allowed to be continuous and only the average transmitted power is upper bounded, based on different channel knowledge assumptions, the capacity of MIMO systems over fading channels is quantified (see e.g. [1–4]). Furthermore, the channel capacity of the continuous MIMO channel with maximum-likelihood equalizer (MLE) or zero-forcing equalizer (ZFE) is shown to be determined by the orthogonal deficiency (OD) of the channel matrix [5]. [6] discusses the MI between the discrete transmitted signal and the continuous equalized signal given by MLE or ZFE. However in practical systems, the transmitted signals are always chosen from some finite alphabet. To recover the transmitted signal, the final detected signals have to be quantified to the same alphabet of the transmitted signals. Therefore, the MI between the discrete transmitted signal and discrete detected signal given by different detectors is of more interest from both the theoretical and practical points of view. However, none of the existing results may be applied to this quantized signal.

In this paper, we focus our study on the practical MI of the discrete MIMO channels with two widely-use detectors, one is the maximum-likelihood detector (MLD) and the other is the zero-forcing detector (ZFD). We need to emphasize that the MI which we consider in this paper is the MI between two specific discrete vectors, i.e., one vector is the transmitted discrete vector, the other is the hard detected vector given by MLD or ZFD. We first establish the definition and analytical expression of the PMI with MLD and ZFD. Next, we study how to quantify the PMI with MLD and ZFD. An easily computable approximation of the PMI with MLD or ZFD is provided which matches well with the simulation results in the mid to high SNR region. Since the quantization process is non-reversible, we deduce that the MI is reduced compared to the MI without quantization. The conclusion is confirmed by both the simulation results and the theoretic results. Consequently, the ultimate reliable data rates that can be transmitted through the MIMO channels with MLD or ZFD are actually less than the values we know before.
2. SYSTEM MODEL

We focus on the discrete-time MIMO channel with $M$ transmitters and $N$ receivers. The transmission over the channel can be described by

$$y = Hs + \omega,$$

where $y$ is the $M \times 1$ received vector, $s$ is the $N \times 1$ transmitted signal, $H$ is an $M \times N$ complex Gaussian distributed channel matrix, and $\omega$ is independent and identically distributed ($i.i.d.$) zero-mean complex Gaussian noise with covariance matrix $E[\omega\omega^H] = \sigma_\omega^2 I_N$. In the following we focus on the case $M = N$.

Let $s_k$ be the $k$-th element of $s$, and is drawn from two-dimensional QAM or one-dimensional PAM constellation $\mathcal{S}$ with equal probability. The SNR of the channel is defined by $\text{SNR} = \frac{E_s}{N_0}$, where $E_s$ is $E[|s_k|^2]$ and $N_0 = 2\sigma_\omega^2$.

2.1. ZFD description

We define the detection ZFD as the whole process including equalization of the received vector $y$ and the quantization of the equalized vector. The first step is to equalize the received vector $y$ as

$$x = H^\dagger y = s + \eta$$

where $H^\dagger = (H^H H)^{-1} H^H$, $x$ is the equalized vector given $y$, and $\eta = H^\dagger \omega$. The covariance matrix of the equalized noise $\eta$ is

$$W = E[\eta\eta^H] = H^\dagger (H^H)^H E[\omega\omega^H] = \sigma_\omega^2 (H^H H)^{-1}.$$

As shown in (3), the equalization introduces the correlation between the noise entries. The ZFD quantizes $x$ symbol by symbol. The $k$th entry of $x$ denoted as $x_k$ is mapped to the alphabet $\mathcal{S}$ as

$$\hat{s}_k = \Omega_s(x_k) = \arg \min_{\hat{s}_k \in \mathcal{S}} |x_k - \hat{s}_k|^2$$

where $\hat{s}_k$ is the $k$-th symbol in $\hat{s}_{ZF}$.

2.2. MLD description

The MLD directly quantizes the received vector $y$ by mapping it to $\mathcal{S}^M$, which is given by

$$\hat{s}_{ML} = \Omega_v(y) = \arg \min_{\hat{s} \in \mathcal{S}^M} \|y - H\hat{s}\|^2$$

where $\Omega_v$ represents the quantization operation of MLD. To be consistent with ZFD, the equivalent quantization of (5) is given as

$$\hat{s}_{ML} = \Omega_v(x) = \arg \min_{\hat{s} \in \mathcal{S}^M} (x - \hat{s})^H W^{-1} (x - \hat{s}).$$

The goal is to estimate the MI from $s$ to $\hat{s}$ with ZFD or MLD.

3. PMI APPROXIMATION OF MLD OR ZFD

We assume perfect channel knowledge is available at the receiver and the input symbols are uniformly distributed over the finite symbol alphabet $\mathcal{S}$.

3.1. PMI Approximation of MLD

Given (5) or (6), we obtain the PMI of MIMO with MLD as follows.

$$I(s; \hat{s}_{ML}|H) = \mathcal{H}(s) - \mathcal{H}(\hat{s}|\hat{s}_{ML}, H)$$

$$= M \log_2 q + \frac{1}{q^M} \sum_{s_v \in \mathcal{S}^M} \sum_{s_u \in \mathcal{S}^M} (P(\hat{s}_{ML} = s_v | s = s_u, H)$$

$$\log_2 \frac{P(\hat{s}_{ML} = s_v | s = s_u, H)}{P(\hat{s}_{ML} = s_v | s = \hat{s}, H)}$$

$$= \log_2 \left(\frac{Q(\sigma_{ML}(uv))}{Q(\sigma_{ML}(v)^2)} + 1 - \sum_{s_u, v \neq u} Q(\sigma_{ML}(uv)^2) \right) \right),$$

(7)

where $q$ is the cardinality of $\mathcal{S}$.

It is difficult to analytically compute (7). However, we find an approximation of the PMI with MLD as follows.

Result 1 (PMI approximation of MLD): Given the system model in (1), the PMI of MIMO systems with the MLD can be approximated by

$$I(s; \hat{s}_{ML}|H) \approx M \log_2 q + \frac{1}{q^M} \sum_{s_v} \left[ \sum_{s_u, u \neq v} Q(\frac{\sigma_{ML}(uv)}{2}) \right.$$

$$\cdot \log_2 Q(\frac{\sigma_{ML}(uv)}{2}) + \left(1 - \sum_{s_u, v \neq u} Q(\frac{\sigma_{ML}(uv)}{2}) \right)$$

$$\cdot \log_2 \left(1 - \sum_{s_u, u \neq v} Q(\frac{\sigma_{ML}(uv)}{2}) \right)\right],$$

(8)

where $Q(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) dt$, $\mu_{ML}(uv) = ||s_u - s_v||_{W^{-1}}^2$ and $\sigma_{ML}(uv) = 2||s_u - s_v||_{W^{-1}}^2$.

3.2. PMI Approximation of ZFD

The PMI with ZFD has the same form as (7) except that the detected signal with MLD $\hat{s}_{ML}$ should be replaced with $\hat{s}_{ZF}$. Different from the PMI with MLD, the PMI with ZFD actually can be numerically computed for QAM or PAM symbol constellation. However, we also provide an easy computable approximation of the PMI with ZFD as follows.
Result 2 (PMI approximation of ZFD): Given the system model in (1), the PMI of MIMO systems with ZFD can be approximated by

$$I(s; \hat{s}_{ZF}|H) \approx M \log_2 q + \frac{1}{Q} \sum_{s_u} \left[ \sum_{s_v, u \neq v} \log_2 \left( \frac{\mu_{ZF}(uv)}{\sigma_{ZF}(uv)} \right) + \left(1 - \sum_{s_v, u \neq v} \frac{\mu_{ZF}(uv)}{\sigma_{ZF}(uv)} \right) \right] \log_2 \left(1 - \sum_{s_v, u \neq v} \frac{\mu_{ZF}(uv)}{\sigma_{ZF}(uv)} \right),$$

where $\mu_{ZF}(uv) = \|s_u - s_v\|^2$, $\sigma_{ZF}^2(uv) = 2(s_u - s_v)\mathbf{A}\mathbf{W}\mathbf{A}(s_u - s_v)$ and $\mathbf{A}$ is any non-identity diagonal matrix with all positive diagonal entries.

4. EXACT PMI WITH ZFD

Before we present the analytical formula of the exact PMI with ZFD, we need the following definitions first. Let

$$z = \left[ \text{Re}\{x\} \quad \text{Im}\{x\} \right], \text{ where Re}\{x\}\{\text{Im}\{x\}\} \text{ is the real( imaginary) part of } x.$$ The covariance matrix of $z$ is $\Sigma = E[zz^T]$. Let $\Omega_{ZF} = \text{det}(\mathbf{A})$ denote the decision cell for $\hat{s}_{ZF} = s_v$ and $\nu = [\text{Re}\{s_u\} \quad \text{Im}\{s_v^T\}]^T$. We define a new function as

$$F(s_u, s_v, \Sigma) = \frac{1}{\sqrt{(2\pi)^M \left| \Sigma \right|}} \int_{c_{a, 1}}^{d_{a, 1}} \cdots \int_{c_{a, 2M-1}}^{d_{a, 2M-1}} e^{-\frac{1}{2} \sum_{k=1}^{2M-1} \left( \varphi_k \right)^2} \prod_{k=1}^{2M-1} \left( \varphi_k \right)^{2M-1} d\varphi,$$

where $a_{s_v,k}$ and $b_{s_v,k}$, for $k = 1, \ldots, 2M$, are respectively the smaller and larger value of the $k$-th coordinates of the edge points of $\Omega_{ZF} = s_v$; $c_{s_v,k} = a_{s_v,k} - u_k$ and $d_{s_v,k} = b_{s_v,k} - u_k$.

When QAM or PAM symbol constellation is employed, the PMI with ZFD can be exactly computed out according to the multivariate normal integrals [7].

Proposition 1: When QAM or PAM constellation is employed, the PMI with ZFD is

$$I(s; \hat{s}_{ZF}|H) = M \log_2 q + \frac{1}{Q} \sum_{s_u \in \mathcal{S}_u} \sum_{s_v \in \mathcal{S}_v} \left( F(s_u, s_v, \Sigma) \right)$$

$$\cdot \log_2 \frac{\sum_{s_u \in \mathcal{S}_u} \sum_{s_v \in \mathcal{S}_v} \left( F(s_u, s_v, \Sigma) \right)}{\sum_{s_p \in \mathcal{S}_u} \sum_{s_v \in \mathcal{S}_v} \left( F(s_p, s_v, \Sigma) \right)}.$$
6. CONCLUSION

In summary, this paper is the first time to study the influence of quantization on the mutual information with MLD or ZFD. We first derive the analytical expression of the PMI with MLD or ZFD. Then based on the expressions, the easy computable approximations of them are proposed. Finally the numerical results show that quantization does reduce the MI, which means there exists the reliable information rates loss by the presence of the quantization step of the detectors.

7. REFERENCES


