OFDM PEAK TO AVERAGE POWER RATIO REDUCTION USING SPARSE BIT-PLANE CODING
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ABSTRACT
A novel Peak to Average Power Ratio (PAPR) reduction method leveraging the rare occurrence of large magnitude Orthogonal Frequency Division Multiplexing (OFDM) symbols is proposed. A distinct innovation of this approach is the exploitation of the sparseness structure of a bit-plane binary number representation of time domain OFDM symbols. By encoding the sparsely populated non-zero entries in the binary bit plane matrix, the dynamic range of OFDM symbols to be transmitted can be significantly reduced without increasing bit error rate. Smaller dynamic range of OFDM symbols implies reduced PAPR, and hence better performance. This method incurs no data rate reduction penalty which may occur when side information must be transmitted via some data subcarriers. Simulation results reveal that this proposed new method out-performs current PAPR reduction methods by wide margins.

Keywords: OFDM, Peak-to-Average-Power-Ratio (PAPR), bit plane coding, sparse coding

1. INTRODUCTION
Orthogonal Frequency Division Multiplexing (OFDM) has been widely used in advanced wireless communication systems, such as 802.11n [1], 802.16 [2], and 4G mobile cellular communication systems, including Long Term Evolution (LTE) and Ultra Mobile Broadband (UMB). The popularity of the OFDM technology stems from its robustness against frequency selective fading and impulse noise, high bandwidth efficiency, and relatively simple receiver implementation.

A design challenge of an OFDM transmitter is its relatively high peak to-average power ratio (PAPR). Due to large number of sub-carriers, some time domain OFDM symbols may have very large magnitudes. When such magnitudes exceed the input range of the Digital to Analog Converter (DAC), or the linear dynamic range of the power amplifier (PA), non-linear distortion of the transmitted signal occurs and causes serious performance degradation.

To mitigate the impacts of excessive PAPR in OFDM transceiver design, three approaches have been investigated in the past: clipping, distribution shaping, and companding.

The simplest and straightforward PAPR reduction approach is clipping [3]. That is, to impose a hard limit on the magnitude of each time domain OFDM symbol. Clipping, however, may cause both in-band and out-of-band interference. It has been proposed to mitigate in-band interference using peak windowing [6] at the expense of further power loss. Out-of-band interference can be mitigated by passing the clipped signals through a Low Pass Filter (LPF) [4], [5]. Since LPF may cause the re-growth of peak values, iterative clipping and filtering is required. The computation cost would be rather high.

Distribution shaping is the second approach for PAPR reduction. Examples of this family of approaches include block coding [7], data permutation, Partial Transmit Sequence (PTS) [8], Selected Mapping (SLM) [9], Tone Reservation [10], [11] and Tone Injection [12]. The key idea common to these methods is to modify the distribution of frequency domain OFDM data so as to reduce the probability of large magnitudes of time domain OFDM symbols. However, none of these methods can guarantee a complete elimination of excessively large magnitudes of time domain OFDM symbols. Thus, clipping may still be needed. Moreover, some of these methods require very complicated computations such as linear programming [10], [11] making them impractical for low power, real time embedded mobile communications.

A third category of PAPR reduction approach is companding [14] which compresses the magnitudes of time domain OFDM symbols non-linearly in the digital domain to reduce excessive PAPR. Then, the magnitudes of OFDM symbols are corrected at receiving end. Companding can be regarded as a generalized scaling or quantization method. The recovered data will subject to varying amount of quantization noise depending on the nonlinear companding function used. Along this direction, a block scaling [13] method has been proposed recently. With block scaling, the set of time domain OFDM symbols are grouped into one block. If any OFDM symbol in the block exceeds a preset threshold, a scaling operation is performed to the entire block to lower PAPR to acceptable level. This data-dependent scaling factor, which is a type of side information, then will be encoded in a sacrificed Sub Carrier (SC) for receiver to decode and restore magnitudes of OFDM symbols. Besides quantization error due to scaling, sacrificed SCs also cause data rate reduction.

In this work, we propose a distinctly novel approach: instead of changing the actual values of individual time domain OFDM symbols by above approaches, our method seeks to modify the representation of these OFDM symbols such that (a) the original OFDM symbol values can be recovered perfectly; while (b) the modified OFDM symbol representation will achieve desired PAPR reduction.

Our new approach exploits the sparse structure of a bit plane matrix representation of the time domain OFDM symbols. By efficiently encoding the positions of the most significant bit (MSB) of those OFDM symbols having large magnitudes, one is able to effectively reduce the number of non-zero bit planes. This in turn reduces the dynamic range of all OFDM symbols so that nonlinear distortion caused by excessive PAPR is eliminated.

All information needed to decode the original higher significant bits of large OFDM symbols are hidden inside the encoded OFDM symbols, and no sub-carrier will be
sacrificed to carry any side information. Therefore, performance improvement can be guaranteed.

This proposed sparse bit-plane coding (SBC) approach is motivated by a careful theoretical analysis of the probability distribution of the MSB bit positions of the time domain OFDM symbols. For bit planes with a very low probability of having any MSB bits, these MSB bit positions can be efficiently encoded using a pointer-based sparse matrix data structure encoding. For bit planes that contain numerous 1s without clear structure, sparse bit-plane coding will not be applied, and these bit planes are left unchanged. If the number of bit planes after MSB encoding is still larger than the maximum allowable dynamic range of subsequent analog circuits, lower bit planes will be discarded.

The rest of this paper is organized as follows. In section II, the probability distribution of the MSB bit level positions will be derived. In section III, the SBC algorithm will be presented and its properties analyzed using results derived in section II. In section IV, empirical performance comparison between the proposed method against existing clipping and filtering approach will be presented. In section V, conclusions of MSB coding algorithm will be provided.

2. ANALYSIS OF MAGNITUDES OF OFDM SYMBOLS

2.1 OFDM Wireless Transceivers

Refer to Figure 1, in a typical OFDM transmitter the channel encoder will produce a binary bit stream. This bit stream then will be segmented into bit vectors of equal length. Each bit vector is converted into a complex-valued modulation symbol $s_{nk}$ based on a modulation mapping scheme. In §2.16, the available mapping types are BPSK (1 bit/symbol), QPSK (2 bits/symbol), 16-QAM (4 bits/symbol), and 64-QAM (6 bits/symbol). The real part and the imaginary part of a modulation symbol will be a $b$-bit binary number. So each modulation symbol encodes $2b$ bits of the binary sequence from the channel encoder.

![Figure 1. A simplified block diagram of transmitter in OFDM systems](image)

The maximum magnitude of $A_k$ is different for different modulation schemes. As such, a scaling factor $s$ will be multiplied to $A_k$, so that the normalized modulation symbol $sA_k$ has the same average power for all different modulation mappings.

With OFDM, an $N$-point IFFT will be applied to the normalized modulation symbol set $\{sA_k\}$. The output will be the time-domain OFDM symbols $\{a_k\}; 0 \leq k \leq N-1$:

$$a_k = \sum_{m=0}^{N-1} s \cdot A_k \cdot W_{kn}^m = \sum_{m=0}^{N-1} s \cdot A_k \exp\left(j \frac{2\pi mk}{N}\right) \quad (1)$$

The real and imaginary part will be translated into analog signal by DAC and are amplified by Power Amplifier before entering air channel.

2.2 Bounds and Probability Distribution of Magnitudes of Time Domain OFDM Symbols

The PAPR problem is caused by the presence of large time domain OFDM symbols. Hence, it is important to understand how the magnitudes of the real part and the imaginary part of the time domain OFDM symbols behave. Due to space limitation, a few important results will be presented without proof in this subsection.

**Lemma 1.**

$$|a_k| \leq \sqrt{2} \cdot s \cdot N \cdot (2^b - 1) + \frac{\sqrt{2} \cdot (2^m - 1)}{\sqrt{2^{2b} - 1}} \cdot N \quad (2)$$

Let us denote $K_N(m)$ to be the maximum number of time domain OFDM symbols whose magnitudes are no smaller than $2^m$. $K_N(m)$ provides an upper bound on the maximum number of MSBs in the $m$th bit plane.

**Lemma 2.**

$$K_N(m) = \left\lfloor \frac{3N \cdot (2^m - 1)}{(2^m - 2^m \cdot (2^m - 2^m - 1))} \right\rfloor \quad (3)$$

The values of $K_N(m)$ for $N = 2048$, and 64QAM modulation are listed in table 1 below.

<table>
<thead>
<tr>
<th>$m$</th>
<th>12</th>
<th>11</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_N(m)$</td>
<td>0</td>
<td>2</td>
<td>9</td>
<td>37</td>
<td>149</td>
<td>597</td>
</tr>
</tbody>
</table>

Next, the statistical properties of time domain OFDM symbols are derived.

**Lemma 3.** Both $\text{Re}\{a_k\}$ and $\text{Im}\{a_k\}$ are normal random variables with $E\{\text{Re}\{a_k\}\} = 0$, and $\text{Var}\{\text{Re}\{a_k\}\} = \text{Var}\{\text{Im}\{a_k\}\} = N/2 \pm \sigma^2$.

Now, denote $q(m) = P\{|\text{Re}\{a_k\}| \in [2^m-1, 2^m]\} = P\{|\text{Im}\{a_k\}| \in [2^m-1, 2^m]\}$ to be the probability that the MSB of either the real part or the imaginary part of a time domain OFDM symbol $a_k$ falls on the $m$th bit plane.

**Lemma 4.**

$$q(m) = \int_{-\infty}^{\infty} \frac{1}{\sigma} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \quad (4)$$

Sample values of $q(m)$ for $N = 2048$ is shown in table 2.

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>10</th>
<th>9</th>
<th>8</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q(m)$</td>
<td>0.234×10^{-14}</td>
<td>4.54×10^{-15}</td>
<td>8.92×10^{-15}</td>
<td>9.69×10^{-5}</td>
<td></td>
</tr>
</tbody>
</table>

A plot of $q(m)$ versus $m$ for $0 \leq m \leq 11$ and $N = 2048$ is given in Figure 2 below.

![Figure 2. $q(m)$ versus $m$ for $N = 2048$](image)

3. MSB Encoding for PAPR Reduction

3.1 Binary Coefficient Matrix Representation

Assume that the magnitude of the real part (or the imaginary part) of a time domain OFDM symbol $a_k$ is an unsigned $M$-bit binary number. Such a binary number can be represented by a $M \times 1$ column vector of 0s and 1s with the most significant bit (MSB) at the top and the least
significant bit (LSB) at the bottom. For example, if \( M = 8 \), a decimal number 73 would be represented as a binary column vector

\[
73 = [0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1]^T
\]

Arranging the corresponding binary column vectors of the real part (or the imaginary part) of \( N \) time domain OFDM symbols, one obtains a \( M \times N \) binary matrix \( S \). A sample image of the \( S \) matrix is depicted in Figure 3 below.

**Figure 3. An example of \( S \) matrix for \( N = 64 \), \( M = 7 \)**

Each row of this \( S \) matrix will be called a bit plane. The bit planes will be labeled from 0 to \( M-1 \) with bit-plane \#0 at the bottom row of \( S \) and bit-plane \#\( M-1 \) at the top row of the \( S \) matrix. Thus, an entry of 1 located at bit plane \#\( m \) corresponds to \( 2^m \). An important observation of Fig. 3 is that the higher level bit planes contain very few non-zero entries (dark squares). Indeed, from tables 1 and 2, it has been predicted that the number of 1 entries in higher bit planes (larger value of \( m \) ) is extremely small. This sparseness of most significant bit (MSB) of time domain OFDM symbols is the key to the proposed sparse bit-plane coding (SBC) method. This procedure is best illustrated with an example:

**Example.** Assume that \( N = 2048 \), \( b = 3 \) and both the real and imaginary part of the OFDM symbols are 16-bit signed binary numbers. Thus, the real and imaginary part of time domain OFDM symbols will each be represented by a 16 by 2048 binary matrix. Suppose \( \text{Re}[a_{312}] = 2 \) and \( \text{Re}[a_{312}] = 2^2 \) \( ( \text{terms smaller than } 2^4 ) \), and the real and imaginary parts of all other OFDM symbols are smaller than \( 2^4 \). In order to represent such information, the encoding of the \( S \) matrix of the real part of OFDM symbols will proceed as follows:

a) The first bit plane that contains at least one non-zero bit is \#10. This information is recorded with 4 bits as 1010. Another 4 bits will be reserved to record the last bit plane that is encoded by MSB coding.

b) Bit plane \#10 can have 0, 1..9 MSBs according to table 1, and hence the number of MSBs can be encoded by 4 bits and is shown as 0001. The position of this MSB, \( k = 312 \) will be saved as an 11-bit binary number 00100110111. In our approach, index 1 is represented by 11 zero bits. The sign bit should also be included in encoding process. Thus, it takes 4 + 11 + 1 = 16 bits to encode this bit plane.

c) Similarly, 6 bits and 11 bits are required for the number of MSBs and index 1124 at bit plane \#9, respectively. And the sign bit of a1124 needs one bit to encode. In addition, the bit plane \#9 entry of previously (partially) reported OFDM symbols a312, 0, will be recorded. Since its position is known, no extra bit is needed. Thus, it takes 6 + 11 + 1 + 1 = 19 bits to encode bit plane \#9.

d) At bit plane \#8, the number of MSBs is 0, so 8 zero bits are needed. Additional two bits are required to record the magnitude portion of previous two MSBs in the order of their appearance. In this example, they are 10. The first 1 refers to \( a_{312} \), and the second 0 refers to \( a_{1124} \). Thus, this bit plane will take \( 8 + 2 = 10 \) bits to encode.

e) We assume bit plane \#7 and below will contain many MSBs so encoding process stops at bit plane \#7. Hence, the ending bit plane number is denoted as 1000. Also, the un-encoded sign bits and un-encoded bit planes will be appended at the end of the encoded bit stream.

Now only \( 8 + 16 + 19 + 10 = 53 \) bits are needed to represent bit plane \#10, \#9, and \#8. It is clearly that the required dynamic range is reduced from 12 = 11 + 1 to 9 without affecting accuracy where additional 1 is for sign bits. Similar procedure is also applied to imaginary part of the \( S \) matrix.

### 3.2 SBC PAPR Reduction Algorithm

Based on the example, one may summarize the proposed MSB PAPR reduction algorithm as follows:

**Step 0.** The highest non-zero bit plane is denoted as beginning bit plane \#\( M_b \). MSB encoding is applied to the highest magnitude bit plane \( ( \text{i.e. } \#M_b \) ) first and then to lower level bit planes in descending order until the length of encoded bit plane is larger than the length of original bit plane. The last bit plane where MSB encoding is performed is denoted as ending bit plane \#\( M_e \). Both beginning bit plane and ending bit plane are 4-bit unsigned binary number. Thus, the header consists of 8 bits.

**Step 1.** Starting from beginning bit plane, denote \( n_b(m) \) as the number of MSBs at bit plane \#\( m \). A header of \( \left\lfloor \log_2(K_M(m)+1) \right\rfloor \) bits will be used to encode the number of MSBs in the current bit plane. Use \( n_b(m)(\log_2N+1) \) bits to encode each MSB’s position and sign at the current bit plane. Use \( \sum_{b_{km}} n_b(k) \) to record the magnitude bits at bit plane \#\( m \) of previously reported MSBs. Thus, the number of bits required to encode bit plane \#\( m \) is

\[
MB(m) = \left\lfloor \log_2(K_M(m)+1) \right\rfloor + n_b(m) + \left( 1 + \log_2 N \right) + \sum_{b_{km}} n_b(k)
\]  

**Step 2.** Construct \( S' \) matrix in the following manner:

From bit plane \#0 to the bit plane \#\( M_e-1 \), copy the corresponding entries of the \( S \) matrix. Fill bit plane \#\( M_e \) and above with the MSB encoded bit stream.

### 4. THE SIMULATION RESULT

#### 4.1 Simulation Setup

Simulations are conducted to compare the performance of the proposed MSB coding PAPR reduction method against state-of-the-art clipping method [5]. We assume \( N = 2048 \) and \( b = 3 \) (64-QAM symbol modulation).

Clipping and filtering operation is performed based on the following equation.

\[
a_{\text{clipping}} = \begin{cases} B_k e^{j\phi_k} & \text{if } |B_k| \leq \gamma \\ \gamma e^{j\phi_k} & \text{otherwise} \end{cases}
\]

where \( B_k \) and \( \phi_k \) represent the amplitude and phase of \( a_k \) in polar coordinate, respectively, and \( \gamma = \frac{2^{\text{Dynamic range}} - 1}{2} \) represents the threshold, a magnitude limitation due to insufficient dynamic range.

When the amplitudes of complex OFDM symbols exceed the threshold, the amplitude is replaced by threshold while phase remains the same. To attenuate the out of band interference, filtering is performed after clipping. In each clipping and filtering operation, oversampling factor is set to 4 to compensate performance loss. The number of
iteration of clipping and filtering is set to 4 to attenuate regrowth [5].

Simulations for both MSB coding approach and clipping and filtering approach are conducted based on 801.11 and 802.16 standards, and the procedure is as follows.

a) At transmitter, a binary bit stream is randomly generated where ‘1’ and ‘0’ are equally probable. Then, it is segmented into many 6 bits groups. Each group is translated into a complex data by 64 QAM mapping. The guard band is also setup in simulations.

b) Frequency domain OFDM symbols are generated by multiplying normalization factor to the generated complex data set. In our simulations, normalization factor is \( \frac{1}{\sqrt{42}} \). Then, time domain OFDM symbols are generated by feeding frequency domain OFDM symbols into IFFT.

c) PAPR reduction is performed by applying SBC encoding algorithm (or clipping) to time domain OFDM symbols for different dynamic ranges. In our simulation, channel is assumed to be ideal, that is, there are no channel noise and multi-path effects.

d) At receiver, the restoration of time domain OFDM symbols is performed by applying MSB decoding to the received encoded bit plane matrices. Then, the restored time domain OFDM symbols or the received clipped OFDM symbols are fed into FFT to generate frequency domain OFDM symbols.

e) Finally, the received frequency domain OFDM symbols are fed into 64 QAM de-mapping to generate the received binary bit stream. By comparing binary bit stream at transmitter and receiver, BER can be calculated.

4.2 Performance Comparison

In Figure 4, compared to clipping and filtering approach, MSB coding can do much better when the number of SCs is large. Also, MSB coding can further reduce dynamic range up to 2 bits while a constant BER of \( 10^{-6} \) is achieved. This performance satisfies the requirement of WiMAX standards (i.e. BER = \( 10^{-6} \) after FEC).

![Figure 4. The comparison between MSB coding and clipping and filtering](image)

Alternative way of viewing this results is that a dynamic range of 7 is sufficient to achieve BER of \( 10^{-6} \) for various number of SCs in MSB coding. Also, there is no sign showing that the required dynamic range will increase significantly when the number of SCs further increases. This implies that a constant dynamic range of 7 is enough for systems with arbitrary number of SCs to achieve BER of \( 10^{-6} \).

5. Conclusion

In this paper, we propose a novel approach to tackle high PAPR problem. By treating OFDM symbol values as sparse bit plane matrixes, we transform a peak reduction problem into a sparse bit plane encoding problem. This enables us to apply the proposed MSB coding algorithm to preserve signal power and also fulfill performance requirement while the required dynamic range is small.

The derivation of expected length of encoded bit plane generated by MSB encoding is provided. Besides, the best encoding range can be also determined by given derivations. The effectiveness of the proposed MSB coding algorithm is further verified by simulation results. Comparing to other PAPR reduction approach, benefit of adopting MSB coding algorithm is much more apparent.

6. Reference