DISTRIBUTED ESTIMATION WITH CONSTANT MODULUS SIGNALS OVER MULTIPLE ACCESS CHANNELS

Cihan Tepedelenlioğlu, Adarsh B. Narasimhamurthy

Arizona State University, Department of Electrical Engineering, Tempe, AZ 85287-5706
{cihan@asu.edu, adarsh.murthy@asu.edu}

ABSTRACT
A distributed estimation scheme where the sensors transmit with constant modulus signals over a multiple access channel is considered. The proposed estimator is shown to be strongly consistent for any sensing noise distribution in the i.i.d. case. When the distributions of the sensing noise are not identical, the existence of a bound on the variances is shown to establish strong consistency. The estimator is shown to be asymptotically normal with an asymptotic variance (AsV) that depends on the characteristic function of the sensing noise. Optimization of the AsV is considered with respect to a transmission phase parameter, and simulations are shown to corroborate our analytical results.

Index Terms— Distributed Estimation, Multiple Access Channel, Constant Modulus, Empirical Characteristic Function

1. INTRODUCTION
In inference-based wireless sensor networks (WSN), low-power sensors with limited battery and peak-power capabilities transmit their observations to a fusion center (FC) for detection or estimation of phenomena. For distributed estimation, much of the literature has focused on a set of orthogonal (parallel) fading channels between the sensors and the FC (please see [1] and the references therein). The bandwidth requirements of such an orthogonal WSN scales linearly with the number of sensors. In contrast, over multiple access channels where the sensor transmissions are simultaneous and in the same frequency band, the utilized bandwidth does not depend on the number of sensors.

The literature on distributed estimation over multiple access channels has mainly involved analog sensor transmission schemes similar to amplify and forward where the instantaneous transmit power is influenced by the sensor measurement noise and is not bounded [2, 3]. The dependence of the transmit power on the sensing noise creates inevitable sensor power management issues, especially when the sensing noise is impulsive.

In this work, for the first time in the literature, a distributed estimation scheme over multiple access channels where the sensors have constant modulus transmissions with fixed instantaneous transmit power is considered. The estimators are universal in the sense of [4] (or “distribution-free” in statistical parlance) in that the estimator does not depend on the distribution or the parameters of the sensing or channel noise. However, unlike the orthogonal framework in [4], multiple access channels are considered herein, and the sensing noise is not assumed bounded.

The sensors transmit with constant modulus transmissions whose phase is linear with the sensed data. The FC estimates the common median of the sensed signal where the sensing noise samples are not assumed to be identically distributed, or from any specific distribution. It is shown that the proposed estimator is strongly consistent under general conditions. While the estimator is shown to be consistent in this general framework, the asymptotic variance of the estimator is derived for the i.i.d. sensing noise and shown to depend on its characteristic function (CF). The optimization of the asymptotic variance with the transmit phase parameter \( \omega \) is considered for different distributions on the sensing noise.

2. SYSTEM MODEL
Consider the sensing model, with \( L \) sensors,

\[
x_i = \theta + \eta_i, \quad i = 1, \ldots, L
\]

where \( \theta \) is an unknown real-valued parameter in a finite interval \( [0, \theta_R] \) of known length, \( \theta_R < \infty \), \( \eta_i \) are a mutually independent, symmetric real-valued noise with zero median (i.e., its pdf is symmetric about zero), and \( x_i \) is the measurement at the \( i^{th} \) sensor. Note that \( \eta_i \) are not necessarily identically distributed, bounded, or have finite moments. We consider a setting where the \( i^{th} \) sensor transmits its measurement using a constant modulus signal \( \sqrt{\rho} e^{j \omega x_i} \) over a Gaussian multiple access channel so that the received signal at the fusion center (FC) is given by

\[
y_L = \sqrt{\rho} \sum_{i=1}^{L} e^{j \omega x_i} + v
\]
optimized, and \( v \) is additive noise. Note that the restriction \( \omega \in (0, 2\pi / \theta R] \) is necessary even in the absence of sensing and channel noise, to uniquely determine \( \theta \) from \( y_L \). The transmitted signal has a deterministic fixed power \( \rho \) which does not suffer from the problems of random transmit power seen in amplify and forward schemes where the transmitted signal from the \( i^{th} \) sensor is given by \( \alpha x_i = \alpha (\theta + \eta_i) \) with instantaneous power per sensor \( \alpha^2 (\theta + \eta_i)^2 \), which is an unbounded random variable (RV) when \( \eta_i \) is. Here \( \alpha \) is a coefficient which might depend on the sensor index, as well as on \( L \) through a power constraint.

The total transmit power from all the sensors in (2) is \( \rho L \). We will consider a fixed per-sensor power constraint case where the total power \( P_T := \rho L \) increases linearly with \( L \). This is in contrast with the fixed total power constraint case where \( \rho = P_T / L \) goes to zero with \( L \) which is considered in [5].

3. THE ESTIMATION PROBLEM

We would like to estimate \( \theta \) from \( y_L \), which is given in (2). We do not necessarily assume that \( \eta_i \) are identically distributed, or that \( \eta_i \) are from any specific distribution since a universal estimator which is independent of the distribution of \( \eta_i \) is desired. Let

\[
\frac{y_L}{L} := e^{i\omega \vartheta L} \sum_{i=1}^L e^{j\omega \eta_i} + \frac{v}{L},
\]

and define \( \varphi_{\eta_i}(\omega) := E [e^{j\omega \eta_i}] \) as the CF of \( \eta_i \). Due to the law of large numbers

\[
\frac{1}{L} \sum_{i=1}^L e^{j\omega \eta_i} \rightarrow \varphi(\omega) := \lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^L \varphi_{\eta_i}(\omega)
\]

almost surely since \( \text{var}(e^{j\omega \eta_i}) = 1 - \sigma_{\eta_i}^2(\omega) \leq 1 \) is bounded [6, (721)]. Whether \( \varphi(\omega) \) is real-valued and whether it is a CF for some distribution will play an important role in the definition and consistency of the proposed estimator. Since \( \eta_i \) are symmetric, \( \{\varphi_{\eta_i}(\omega)\} \) are real-valued and therefore \( \varphi(\omega) \) is also real-valued. Since convex combinations of CFs are CFs, the partial sums \( L^{-1} \sum_{i=1}^L \varphi_{\eta_i}(\omega) \) are as well [7]. However its limit, \( \varphi(\omega) \) in (4), is not necessarily a CF, unless \( \varphi(\omega) \) is continuous at \( \omega = 0 \), by the continuity theorem [7, Corollary 1.2.2].

The natural estimator that we will adopt is based on the phase of \( z_L \), which we express as

\[
\hat{\theta} := \frac{1}{\omega} \tan^{-1} \left( \frac{z^R_L}{z^I_L} \right),
\]

where \( z^R_L := \text{Re}\{z_L\} \) and \( z^I_L := \text{Im}\{z_L\} \). Note that this estimator does not depend on the statistics of \( \eta \) or \( v \), as desired. We now establish the strong consistency of the proposed estimator \( \hat{\theta} \):

**Theorem 1.** The estimator \( \hat{\theta} \) in (5) is strongly consistent provided that \( \omega \in (0, 2\pi / \theta R] \) is chosen to satisfy \( \varphi(\omega) \neq 0 \).

**Proof.** Since \( \eta_i \) are i.i.d., due to the strong law of large numbers \( z^R_L \rightarrow \varphi(\omega) = \sqrt{\rho} \cos(\omega \theta) \varphi(\omega) \) and \( z^I_L \rightarrow \varphi(\omega) = \sqrt{\rho} \sin(\omega \theta) \cdot \varphi(\omega) \) almost surely. Since \( \hat{\theta} \) in (5) is a continuous function of \( \left[ z^R_L \right. \left. z^I_L \right] \), \( \hat{\theta} \to (1/\omega) \tan^{-1} (\varphi(\omega) / \varphi(\omega)) = 0 \) almost surely [8, Thm 3.14]. We need the assumption that \( \varphi(\omega) \neq 0 \) since otherwise \( \theta \) cannot be uniquely determined from \( \varphi(\omega) \).

We now investigate when an \( \omega \) that satisfies the conditions of Theorem 1 exists thereby guaranteeing strong consistency. Consider the identically distributed case where \( \eta_i \) have a common distribution with a RV \( \eta \) so that \( \varphi(\omega) = \varphi_\eta(\omega) \) is a CF. Many distributions such as Gaussian, Laplace, and Cauchy satisfy \( \varphi_\eta(\omega) > 0 \) for all \( \omega \). If the common sensing noise distribution is known to have this property, then any choice of \( \omega \) would clearly satisfy the conditions of Theorem 1. In the more general case, where nothing is known or assumed about \( \eta \) it can be shown that \( \omega \) can be chosen close enough to the origin because all CF at the origin are equal to 1 and continuous, and therefore there is a neighborhood around \( \omega = 0 \) for which \( \varphi(\omega) > 0 \). So, for the identically distributed sensing noise, an \( \omega \) satisfying the conditions of Theorem 1 can be found even if the sensing noise variance does not exist.

In the general non-identically distributed case, this argument does not follow since \( \varphi(\omega) \) is not necessarily a CF. However, if \( \varphi(\omega) \) is continuous at \( \omega = 0 \), then since \( \varphi(\omega) \) is the limit of a sequence of CFs, it will be a CF by the continuity theorem [7] and the argument above follows. For an example of a case where \( \varphi(\omega) \) is not a CF and not continuous at \( \omega = 0 \), consider a case where \( \sum_{i=1}^L \varphi_{\eta_i}(\omega) = \infty \) for all \( \omega > 0 \) such as when \( \eta_i \) are Gaussian with variances that depend on \( i \) linearly: \( \varphi_{\eta_i}(\omega) = e^{-\sigma_{\eta_i}^2 \omega^2/2} \) where \( \sigma_{\eta_i}^2 = \sigma^2 \). In this case due to the \( L^{-1} \) factor in (4), \( \varphi(\omega) = 0 \) when \( \omega > 0 \), and \( \varphi(0) = 1 \). For this example, \( \varphi(\omega) \) is not a CF for any distribution, and there exists no \( \omega \) that satisfies the requirements of Theorem 1. Clearly, this is a very severe case where the sensing noise variance increases linearly with the sensor index, without bound. In the next theorem, we show that such cases can be ruled out if the variances of \( \eta_i \) exist and are bounded uniformly in \( i \).

**Theorem 2.** Let \( \varphi(\eta_i) \) exist for all \( i \), and

\[
\sigma_{\eta_i}^2 := \text{sup} \varphi(\eta_i) < \infty
\]

then any \( 0 < \omega < \min \left( 2\pi / \theta R, \sqrt{2} / \sigma_{\eta_i}^2 \right) \) satisfies \( \varphi(\omega) > 0 \), thereby fulfilling the requirement of Theorem 1 on \( \omega \).

**Proof.** From [7, pp. 89] we have \( \varphi_{\eta_i}(\omega) \geq 1 - \sigma_{\eta_i}^2 \omega^2 / 2 \). Using (4) we have \( \varphi(\omega) \geq 1 - \lim_{L \to \infty} L^{-1} \sum_{i=1}^L \varphi_{\eta_i}(\omega) \geq 1 - \sigma_{\max}^2 \omega^2 / 2 \geq 0 \) where the last inequality holds provided that \( \omega < \sqrt{2} / \sigma_{\max} \). Since also \( \omega \leq 2\pi / \theta R \) we have the theorem.
4. PERFORMANCE ANALYSIS

The estimator in (5) relies on constant modulus transmissions from the sensors to the FC, and is strongly consistent over a wide range of scenarios outlined above. However, the performance of \( \hat{\theta} \) will depend on statistical assumptions on \( \{ \eta_i \} \) and \( v \). The following theorem characterizes this performance, under the assumption that \( v \sim CN(0, \sigma_v^2) \) and \( \{ \eta_i \} \) are identically distributed with an arbitrary common distribution.

**Theorem 3.** \( \sqrt{T} (\hat{\theta} - \theta) \) is asymptotically normal with zero mean and variance given by,

\[
AsV(\omega) = \frac{1 - \varphi_\eta(2\omega)}{2\omega^2 \varphi_\eta^2(\omega)}
\]

**Proof.** Please see [5]. \( \square \)

Note that in the i.i.d. case (3) is the empirical characteristic function (ECF) [7] of \( \eta_i \) corrupted by additive noise. While the ECF has been studied extensively in the statistical literature for constructing centralized estimators (please see [7] and references therein) it has not been addressed in the context of communication of samples as in distributed estimation, and therefore issues of power constraint and channel noise have not arisen in the literature on parameter estimation with ECFs.

The proposed estimator is consistent under general conditions and does not depend on the noise parameters. However, if the noise distribution and parameters are available, it is possible to minimize the \( AsV \) with respect to \( \omega \) over the interval \((0, 2\pi/\theta_R] \):

\[
AsV^* = \inf_{\omega \in (0, 2\pi/\theta_R]} \frac{1 - \varphi_\eta(2\omega)}{2\omega^2 \varphi_\eta^2(\omega)}
\]

We now consider solving (7), and investigate the behavior of \( AsV(\omega) \) near the origin to see under what conditions small \( \omega \) will yield optimum performance. Using l’Hôpital’s rule, it is seen that \( \lim_{\omega \to 0} AsV(\omega) = \sigma_\eta^2 \) the variance of \( \eta \), when \( \eta \) has finite variance. In fact, when also the fourth moment \( \mu_4 \) of \( \eta \) exists, we have a stronger result:

**Theorem 4.** If the first four moments of \( \eta \) exists, then \( AsV(\omega) \) in equation (7) satisfies

\[
AsV(\omega) = \sigma_\eta^2 - \frac{1}{3} \kappa_\eta \sigma_\eta^4 \omega^2 + o(\omega^2)
\]

as \( \omega \to 0 \), where \( \kappa_\eta = \mu_4 / \sigma_\eta^4 - 3 \) is the excess kurtosis of \( \eta \).

**Proof.** We have already established that the first term in \( \phi(\omega) \) and \( \sigma_\eta^2 \). Using the Maclaurin series expansion of \( \varphi_\eta(\omega) \) in terms of the second and fourth moments of \( \eta \), the numerator and denominator of (7) can be expressed as \( N(\omega) = 2\sigma_\eta^2 \omega^2 + (2/3) \cdot \mu_4 \omega^4 + o(\omega^4) \) and \( D(\omega) = 2\omega^2 (1 - (1/2)\sigma_\eta^4 \omega^2 + (\mu_4/24)\omega^4 + o(\omega^4)) \), respectively. By taking the second derivative and evaluating we have

\[
\frac{\partial^2 N(\omega)}{\partial \omega^2} D(\omega) \bigg|_{\omega=0} = -2 \rho_4 + 2 \rho_4^4
\]

Dividing by \( 2! \) we obtain the coefficient of \( \omega^2 \) in the Maclaurin series, as given in (8).

**Theorem 5.** If \( \eta \) is Gaussian then the best asymptotic performance for \( \hat{\theta} \) in (5) for the per-sensor power constraint satisfies \( AsV^* = \sigma_\eta^2 \).

**Proof.** Equation (8) shows that \( \lim_{\omega \to 0} AsV(\omega) = \sigma_\eta^2 \) which implies that \( AsV^* \leq \sigma_\eta^2 \). To see that \( AsV^* \geq \sigma_\eta^2 \) consider a benchmark genie-aided sample mean estimator \( \hat{\theta}_{GA} = L^{-1} \cdot \sum_{i=1}^L x_i \), which has access to the sensor measurements \( \{ x_i \}_{i=1}^L \), rather than \( y_L \) in (2). The sample mean which has an asymptotic variance of \( \sigma_\eta^2 \), achieves the Cramer Rao bound (CRB) for an estimator of \( \theta \) from \( \{ x_i \}_{i=1}^L \) since \( \eta \) is Gaussian by assumption. Since \( \theta \to \{ x_i \}_{i=1}^L \to z_L \) forms a Markov chain, from the data processing inequality for the CRB [9], the CRB for estimators based on \( z_L \) is at least that obtained for the genie-aided setup of estimating \( \theta \) from \( \{ x_i \}_{i=1}^L \), which is \( \sigma_\eta^2 \). Therefore, the best achievable performance in the per-sensor power case cannot be better than that of \( \hat{\theta}_{GA} \), which implies \( AsV^* \geq \sigma_\eta^2 \). \( \square \)

Note that in the proof of Theorem 5 we used the Gaussianity only to assert that the sample mean achieves the CRB. Theorem 5 also holds for any other distribution with this property. To conclude, small \( \omega \) yields good asymptotic performance which does not depend on the a priori range \( \theta_R \). The performance can be improved by appropriately increasing \( \omega \) in the neighborhood of \( \omega = 0 \) when \( \eta \) is from an impulse distribution with positive excess kurtosis.

5. SIMULATIONS

In this section, we illustrate the behavior of \( L \cdot \text{var}(\theta - \hat{\theta}) \) with respect to \( \omega \) and \( L \) for various noise distributions. In Fig.1, we consider a system with \( L=500, \sigma_\eta^2=0.5, \sigma_\eta^4=1 \) and \( \rho=1 \). For all noise distributions considered, for finite \( L \), it is seen that values of \( \omega \) near 0 leads to large \( L \cdot \text{var}(\theta - \hat{\theta}) \). For Gaussian distributed sensing noise, \( AsV(\omega) \) is seen to have a minimum value of \( \sigma_\eta^2 \), as indicated in our analysis. For Laplace
distributed sensing noise, for values of ω near 0 a very similar behavior as seen for Gaussian sensing noise is observed in Fig.1. It is also seen that the minimum $A\bar{s}V(\omega)=0.375 < \sigma^2_\eta$ for Laplace distribution is attained at $\omega \approx 1.4$ unlike the Gaussian case. For $\omega \geq 0.3$ it is observed that $L \cdot \text{var}(\theta - \hat{\theta})$ matches $A\bar{s}V(\omega)$ for both Gaussian and Laplace distributed sensing noise, thereby corroborating our results. Since the variance of Cauchy distribution is undefined, for $\omega \approx 0$, both $L \cdot \text{var}(\theta - \hat{\theta})$ and $A\bar{s}V(\omega)$ are very large for Cauchy distributed sensing noise. But, unlike the Gaussian and Laplace distributions, $L \cdot \text{var}(\theta - \hat{\theta})$ matches $A\bar{s}V(\omega)$ for all values of ω.

In Fig.2, we show that $L \cdot \text{var}(\theta - \hat{\theta})$ converges to $A\bar{s}V(\omega)$ for the various sensing noise distributions. Here we consider a system with $\sigma^2_\eta=0.5$, $\sigma^2_v=1$, and $\rho=1$. For all sensing noise distributions considered, the optimal value of ω is obtained from Fig.1. In Fig.2, it is seen that $L \cdot \text{var}(\theta - \hat{\theta})$ for Gaussian distributed sensing noise converges to $A\bar{s}V(\omega)=0.5$ for values of $L \geq 140$. But for Laplace distributed sensing noise, convergence to $A\bar{s}V(\omega)=0.375$ occurs around $L \approx 80$ and for Cauchy distributed sensing noise, convergence to $A\bar{s}V(\omega)=0.52$ is seen at about $L \approx 60$. Additionally, for Cauchy distribution, it is seen that $L \cdot \text{var}(\theta - \hat{\theta})$ increases with $L$ initially before converging to $A\bar{s}V(\omega)$. This is due to the impulsive nature of the noise distribution.

6. CONCLUSIONS

A distributed estimation scheme over multiple access channels with constant modulus signaling is proposed. Its strong consistency and asymptotic normality is established. The $A\bar{s}V$ is shown to depend on the characteristic function of the sensing noise distribution evaluated at the transmission phase parameter ω. It is shown that the $A\bar{s}V$ can be minimized for small values of ω for the Gaussian sensing noise case. The minimizing ω and the resulting $A\bar{s}V$ do not depend on the range parameter $\theta_R$. When the sensing noise is impulsive with a positive excess kurtosis, then the $A\bar{s}V$ can be improved by choosing a larger ω in which case the performance depends on $\theta_R$.

7. REFERENCES