Abstract—Wireless source localization has found a number of applications in wireless sensor networks. In this work, we investigate source localization based on the practical time of arrival (TOA) measurement model. Unlike most existing works that transform TOA measurement into time differences before processing, we consider the original measurement model and investigate three methods for direct source localization. We also derive the Cramér-Rao lower bound (CRLB) under the TOA model and establish its connection with the CRLB under the more commonly used time-difference of arrival (TDOA) signal model. We present results that illustrate the performance advantage of source localization based on the original TOA model over the commonly used TDOA pre-processing.

I. INTRODUCTION

With the recent surge in interest and applications of wireless sensor networks, the problem of source localization has become increasingly important [1]. Wireless source localization has found broad application in target tracking, signal routing, interference cancellation, and emergency response, among others. Source localization involves estimating positions of signal emitters in a network of sensors that measure source signal features. A data fusion center then estimates the source location(s) from the collection of sensor signal measurements. The various measurement methods include measuring source signals’ time of arrival (TOA), time difference of arrival (TDOA), received signal strength (RSS), angle of arrival (AOA), and their combinations [2].

It should be noted that, in many applications, direct distance and signal strength measurements may not be available for source estimation. Additionally, in a wireless environment with rich scatters, signal strength measurement can be highly difficult to model and noisy. Therefore, alternative measurement models may be more practical. In particular, we are interested in the simple measurement model based on signal arriving time. In the TOA model, each sensor only needs to identify a special signal feature such as a known preamble or postamble string and record its time of arrival. Without knowing the exact time of the signal transmission \( t_0 \), the TOA measurement is also prone to measurement error because of clock synchronization errors at various sensors and because of scatters and multipath effects. One common approach in the literature to deal with the unknown \( t_0 \) is to pre-process the TOA measurement by considering only the difference of TOA measurements from various sensors. This subtraction pre-processing removes the unknown \( t_0 \) from the measurement and simplifies the source localization problem. The time-difference of arrival model considers the difference of the arrive time between the anchor nodes, which avoids the problem of synchronization. However, in order to get the TDOA measurement, one TOA measurement is subtracted from another, which makes the noise signals corrupting different TDOA measurements correlated [2]. The drawbacks of such pre-processing arise from the fact that measurement subtraction leads to noise correlation and noise amplification (by 3dB). Despite the drawbacks due to pre-processing in obtaining TDOA measurement, the simplicity of the TDOA model has spurred a number of investigations of TDOA measurements for source localization including linear [2], nonlinear [3], and convex relaxation [4] approaches. In fact, most such works imprecisely assumes the TDOA to be the original measurement with uncorrelated noises and neglected the correlations introduced by the pre-processing step. As a result, the actual effect of the subtraction pre-processing on the localization performance has not been rigorously assessed.

In this work, we focus on the original measurement model of TOA without pre-processing for source localization estimation. We describe new and robust methods for direct source localization based convex optimization. In particular, we present a semidefinite programming (SDP) relaxation and a second order cone programming (SOCP) algorithms based on the TOA model. We determine the Cramér-Rao lower bound (CRLB) under the TOA model and establish its connection with the CRLB under the commonly used TDOA pre-processing model. We further illustrate the relationship between the proposed TOA algorithm and the associate CRLB.

II. PROBLEM FORMULATION

A. Time of Arrival Measurement Model

Consider a network of \( N \) sensors at known positions that are denoted by a set of \( m \)-dimensional vectors \( \mathbf{x}_1, \ldots, \mathbf{x}_N \) (with \( m = 2 \) or 3). These sensors cooperate to determine an unknown source location denoted by an \( m \)-dimensional vector \( \mathbf{y} \). Note that we restrict our focus only to a propagation environment without remote scatters. In other words, the time of arrival measurement can be approximately modeled by a line of sight environment. By collecting measurements from the sensors, a data fusion center generates an estimate \( \hat{\mathbf{y}} \) of the source location.

The time of arrival measurement \( t_i \) at node \( \mathbf{x}_i \) can be
modeled by
\[ t_i = \frac{1}{c} \| x_i - y \| + t_0 + n_i, \quad i = 1, 2, \cdots, N, \] (1)
where \( c \) is the speed of light, \( t_0 \) is the unknown time of transmission at the source, and \( n_i \) is the independent identically distributed (i.i.d.) Gaussian noise with zero mean and variance \( \sigma^2 \). Under the i.i.d. Gaussian noise assumption, the conditional probability density of the measurement data is
\[ p(t_1, t_2, \ldots, t_N | y, t_0) = (2\pi\sigma^2)^{-N/2} \exp \left( -\frac{1}{\sigma^2} \sum_{i=1}^{N} (t_i - \frac{1}{c} \| x_i - y \| - t_0)^2 \right). \] (2)

**B. Direct MLE Search**

The maximum likelihood estimate (MLE) of \( y \) is
\[ \hat{y} = \arg \min_{y, t_0} \sum_{i=1}^{N} \left( t_i - \frac{1}{c} \| x_i - y \| - t_0 \right)^2. \] (3)

Thus, the MLE requires a joint search of unknown \( y \) and \( t_0 \). As a brute-force approach, the direct optimization can be implemented as the MLE by directly searching for the optimum \( y \) and \( t_0 \). In this paper, we apply the Powell algorithm [5] to get the optimum MLE of the source position.

**C. SDP Relaxation Formulation**

Instead of solving the MLE jointly, we can derive the estimate by maximizing the likelihood function in two steps. First, we can estimate the optimum transmission time \( t_0 \) as a function of the unknown \( y \). In particular, for zero mean noise \( n_i \) in (1), the MLE of transmission time \( t_0 \) is
\[ \hat{t}_0 = \frac{1}{N} \sum_{i=1}^{N} \left( t_i - \frac{1}{c} \| x_i - y \| \right). \] (4)

Next, we substitute \( t_0 \) with the MLE \( \hat{t}_0 \) into the likelihood function or, equivalently, into the objective function (3) before finding the optimum source location \( y \) by maximizing the likelihood function.

We first introduce some auxiliary variables \( \tau_i = \| x_i - y \| / c \). Let \( \tau = [\tau_1, \ldots, \tau_N]^T \), \( A = [1, 1, \ldots, 1]^T \), \( G = I - \frac{1}{c} AA^T \), and \( Q = \tau \tau^T \). After substituting \( t_0 \) with \( \hat{t}_0 \), we can rewrite the objective function in (3) as
\[ \min_{y, \tau_0, \tau} \operatorname{Tr}(Q^T G (Q - \tau \tau^T - t \tau^T + t t^T)) \] (5)
The constraint \( t_0 = \frac{1}{c} \| x_i - y \| \) can be expressed as
\[ Q_{ii} = \tau_i^2 = \frac{1}{c^2} \| x_i - y \|^2 \] (6)
\[ = \frac{1}{c^2} \left[ x_i - y \right]^T \left[ \begin{array}{cc} I & y \end{array} \right] \left[ \begin{array}{c} x_i \\ y \end{array} \right], \]
where \( y_s = y^T y. \) Moreover, from the Cauchy-Schwartz inequality, we have
\[ Q_{ij} = \tau_i \tau_j = \frac{1}{c^2} \| x_i - y \| \| x_j - y \| \] \[ \geq \frac{1}{c^2} \left[ x_i - y \right]^T \left[ \begin{array}{cc} I & y \end{array} \right] \left[ \begin{array}{c} x_j \\ y \end{array} \right], \] (7)
Then we can relax the constrained minimization problem into the following SDP form
\[ \min_{y, \tau_0, \tau, Q} \operatorname{Tr}(Q^T G (Q - \tau \tau^T - t \tau^T + t t^T)) + \eta \sum_{i=1}^{N} \sum_{j=1}^{N} Q_{ij} \] (8)
\[ \text{s.t.} \quad \left[ \begin{array}{cc} Q & \tau \\ \tau^T & 1 \end{array} \right] \geq 0, \left[ \begin{array}{cc} I & y \\ y^T & y_s \end{array} \right] \geq 0, \] (9)
\[ Q_{ii} = \frac{1}{c^2} \left[ x_i - y \right]^T \left[ \begin{array}{c} I \\ y \end{array} \right] \left[ \begin{array}{c} x_i \\ y_s \end{array} \right] = 1, \] (10)
\[ Q_{ij} = \frac{1}{c^2} \left[ x_i - y \right]^T \left[ \begin{array}{c} I \\ y \end{array} \right] \left[ \begin{array}{c} x_j \\ y_s \end{array} \right] = 1, \] (11)
\[ \eta > 0 \text{ is a penalty factor designed to enforce the constraint } \tau_i = \| x_i - y \| / c. \] The convex optimization problem of (8) can be solved efficiently using known solvers such as SeDuMi [6].

**D. SOCP Relaxation Formulation**

Note that the full MLE of \( y \) in (3) is not convex. The problem of (3) can be equivalently written as
\[ \min_{y, t_0, \tau_0, \tau} \sum_{i=1}^{N} (t_i - \tau_i - t_0)^2 \] (9)
\[ \text{s.t.} \quad \tau_i = \frac{1}{c} \| x_i - y \|, \quad i = 1, \ldots, N. \]

We can relax the equality constraints to inequality constraints that are convex. In addition, we need to add a penalty term to the objective function to avoid meaningless solutions (with large \( \tau_i \) and small \( t_0 \)). By adding a penalty factor \( \eta > 0 \), we obtain the following SOCP formulation
\[ \min_{y, t_0, \tau_0, \tau} \sum_{i=1}^{N} (t_i - \tau_i - t_0)^2 + \eta \sum_{i=1}^{N} \tau_i^2 \] (10)
\[ \text{s.t.} \quad \| x_i - y \| \leq c \cdot \tau_i, \quad i = 1, 2, \ldots, N, \]
where \( \eta \) is a penalty factor. Once again, the optimal solution of this convex optimization problem can be found using modern SOCP solvers such as SeDuMi [6].

**III. CRAMÉR-RAO LOWER BOUND**

In this section, we investigate the CRB of the TOA measurement model. We derive the CRB of the location estimation under TOA model and compare it against the more common TDOA model.

Let \( y_k, x_{ik} \) denote the \( k \)-th elements of vectors \( y, x_i \), respectively. We can now determine the Crámer-Rao lower bound (CRB) of the source location estimate. Let \( z = [y_1, \ldots, y_m, t_0]^T \) form the vector of all unknowns. We are interested in the CRB of \( z \). From (2), we have the \((k, j)\)-th element of the Fisher information matrix \( F_a \) as
\[ [F_a]_{kl} = \frac{1}{c^2 \sigma^2} \sum_{i=1}^{N} \frac{(y_k - x_{ik})(y_j - x_{ij})}{\| x_i - y \|^2}, \] (11)
where \( 1 \leq k \leq l \leq m. \) Additionally, we have
\[ [F_a]_{k(m+1)} = [F_a]_{(m+1)k} = \frac{1}{c^2 \sigma^2} \sum_{i=1}^{N} \frac{y_k - x_{ik}}{\| x_i - y \|^2}, \] (12)
where \( 1 \leq k \leq m \) and \([\mathbf{F}_a]_{(m+1)(m+1)} = \frac{N}{\hat{z}}\).

Hence, the CRLB of the estimate \( \hat{y} \) is

\[
\text{MSE}_\text{TOA} = E\left(\|\hat{y} - y\|^2\right) \geq \sum_{i=1}^{m} \mathbf{F}^{-1}_{ii} \cdot (13)
\]

Since the term \( \nabla_z \ln(p(t_1, ..., t_N|z)) \) does not always equal to zero, we find that there exists no unbiased efficient estimator for \( y \). Thus, the MLE is not efficient and no unbiased estimate can achieve the CRLB under the TOA model (1). This conclusion is consistent with the proof in [7] because \( t_i \) is not linearly independent since they have \( n_i \) in common. Therefore, by preprocessing through subtraction, no TDOA algorithm would perform better than the direct MLE search algorithm.

Unfortunately, some existing source location approaches based on TDOA measurements neglect the pre-processing needed to obtain measurement \( \Delta_{ij} \). In particular, the correlation of the noise term in \( \Delta_{ij} \) is ignored. By assuming the TDOA measurement noise as independent and Gaussian, the \( N(N - 1)/2 \) pairwise TDOA measurements \( \Delta_{ij} (i = 1, ..., N, j = i + 1, ..., N) \) follow a joint conditional probability density function of

\[
p(\Delta_{ij}, i = 1, ..., N, j = i + 1, ..., N|y) = \frac{1}{(4\pi\sigma^2)^{N(N-1)/2}} \exp\left(-\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{4\sigma^2}(\|x_i - y\| - \|x_j - y\|)^2 \right) . \quad (15)
\]

As a result, the \((k, l)\)-th element \((1 \leq k \leq l \leq m)\) of this Fisher information matrix is

\[
[F_d]_{kl} = \frac{1}{2\sigma^2} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \left(\frac{x_{ik} - y_i}{\|x_i - y\|} - \frac{x_{jl} - y_j}{\|x_j - y\|}\right) \left(\frac{x_{ik} - y_i}{\|x_i - y\|} - \frac{x_{jl} - y_j}{\|x_j - y\|}\right) \cdot \quad (16)
\]

Without considering the pre-processing of the TOA data, this fictitious TDOA signal model can lead to misleading results. By assuming independent noises in TDOA measurement, every pairwise \( \Delta_{ij} \) is providing useful new information for source localization. As a result, TDOA algorithm designed according to this erroneous model often seems to generate highly optimistic results. Similarly, the CRLB based on \( F_d \) is also optimistic and fails to describe the true lower bound of an unbiased TDOA algorithm.

In practice, if the pre-processing is used, then the CRLB for an unbiased TDOA algorithm can be attained by considering only independent pairwise time differences \( \Delta_{ij} \). By appointing a reference sensor \( r \), we can include \( N - 1 \) TDOA measurements \( \{\Delta_{ir}, i \neq r\} \) in order to derive the corresponding CRLB as in [8]. It is shown that there exists no unbiased efficient estimators when we consider the correlated noise.

IV. SIMULATION RESULTS

In this section, we give two examples to compare the three TOA algorithms with the classic TDOA algorithm [2] (labeled as Classic-TDOA). We will also illustrate the three CRLBs discussed in Section III. We denote the three TOA approaches as Search-TOA, SDP-TOA, and SOCP-TOA, respectively, in the order of their presentation. We also use CRLB-TOA, CRLB-TDOAU, and CRLB-TDOAC, respectively, to represent the CRLB of the TOA model, the TDOA model with uncorrelated noise, and the TDOA model based on subtraction preprocessing. We evaluate the root mean squared error (RMSE) of the source position as the performance measure against different value of the noise standard deviation.

In the first example, we put three anchor nodes at \([400, 0]^T, [-400, 0]^T, [0, -400]^T\). The source is located at \([0, -50]^T\) and \( t_0 \) is randomly set with normal distribution of zero mean and variance 1. The penalty factor \( \eta \) is set to 0.0001 for the SOCP-TOA and SDP-TOA algorithms. In Figure 1, we compare the performance of Classic-TDOA, SOCP-TOA, SDP-TOA, and Search-TOA algorithms. It can be seen that the performance of Classic-TDOA, SDP-TOA, and Search-TOA algorithms are very close to each other and less than 1 dB from the CRLB-TOA. However, the performance of SOCP-TOA approach is about 1 dB worse at the same level of RMSE. Clearly in this case, the number of sensor nodes is small whereas the cone relaxation is too lose. We also note that in this example, the true CRLB of the TOA model is significantly lower than the misleading CRLB obtained using the fictitious TDOA model.

\[\text{Fig. 1. Performance comparison for the three sensor case.}\]

In the second example, we use more sensor nodes by
placing 8 sensors at \([400, 400]^T, [-400, 400]^T, [400, -400]^T, [-400, -400]^T, [800, 800]^T, [800, -800]^T, [-800, 800]^T, [-800, -800]^T\). The source is located at \([10, 0]^T\). We set \(t_0\) to be normally distributed with zero mean and variance 1. The parameter \(\eta\) for SOCP-TOA and SDP-TOA algorithms is set to 0.0005. The performance of the various algorithms along with the bounds is given in Figure 2. From the results, we can see that the performance of both SDP-TOA and Search-TOA (MLE) are very close from low to moderate SNR. At high SNR, the MLE searching leads to slightly better performance. Although the SOCP-TOA performance is not as good, it is still more reliable than the classic linear TDOA algorithm.

It is also interesting to note that all three TOA algorithms outperform the CRLB derived for the TDOA model based on pre-processing. This comparison demonstrates the drawback of pre-processing in the TDOA model. Moreover, we also observe that a significant gap exists between the CRLB and all the tested algorithms. This observation illustrates that there still exists great potential for developing new and better algorithms to improve the source localization accuracy based on TOA measurement.

Finally we test the sensitivity of the proposed SOCP-TOA and SDP-TOA algorithms to the selection of the penalty factor. We fix the noise variance \(\sigma^2\) to 10dB in the second example and compare the RMSE values of \(y\) by applying different values of \(\eta\) in. The resulting performance is given in Figure 3. It can be seen that both algorithms (SOCP-TOA and SDP-TOA) are not very sensitive to the choice of the penalty factor \(\eta\). Our experience shows that \(\eta\) can be chosen between \(10^{-5}\) and \(10^{-3}\) for reliable estimation of \(y\).

V. CONCLUSION

We study the source localization problem based on the practical TOA model in wireless sensor network. We describe three algorithms for estimating the source location based on maximum likelihood estimation and convex optimization relaxations. We also examine the CRLB for source localization under the TOA model. By analyzing the pre-processing effect in the more common TDOA model, we establish the loss of performance due to pre-processing and demonstrate the existence of a common misconception associated with source localization based on the TDOA model. Our results strongly motivate the continued effort to develop more efficient and accurate source localization methods under the TOA model.

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REFERENCES