THE ROLE OF THE AMBIGUITY FUNCTION IN COMPRESSED SENSING RADAR
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ABSTRACT
As is clear from the ambiguity function (AF) uncertainty principle and the underlying matched filtering operation, pulse compression (PC) radars cannot accurately and simultaneously measure both the time delay and Doppler shift of moving targets. On the other hand, compressed sensing (CS) radar would seem to be emerging as a means to do just that, jointly to estimate the delay and Doppler shift, through convex optimization instead of matched filtering, such that PC resolution limits no longer apply. However, it turns out that the AF is closely related to the CS “dictionary coherence coefficient,” which affects the optimization recoverability. In this paper we formalize this relationship.

Index Terms—Radar, compressed sensing, ambiguity function, pulse compression, time-frequency (circular) shift dictionary.

1. INTRODUCTION
Pulse compression (PC) is of wide use in radar systems to increase return energy without an increase in power, via relatively long pulses of sufficient bandwidth that no degradation in range resolution occurs. Assuming an additive white noise environment and a stationary point scatterer there is no better processor than the matched filter; however, a moving target introduces Doppler shift. Doppler mismatch can degrade system performance: e.g., enlarging the sidelobe level, decreasing the SNR, and reducing the range resolution [8, 5]. As a result, the attractive properties of PC may no longer hold.

Current methods to deal with Doppler effects in radar signal processing include: (1) suppression, (2) compensation, and (3) utilization. Suppression involves the design of Doppler insensitive waveforms such as linear frequency modulation (LFM) and “mismatched” filters [10] to alleviate the interference of moderate frequency variation [8, 5]. Compensation refers to the use of prior knowledge or Doppler estimation to revise the received signal prior to PC. The first two are based on matched filtering, and due to the uncertainty principle of AFs, the radar system cannot accurately measure both the time delay and Doppler shift [4, 8]. Compressed sensing (CS) radar is on the other hand a sort of utilization: it tries jointly to estimate the delay and Doppler of multiple targets using convex optimization rather than matched filtering. Since the delay and Doppler are completely decoupled in their estimation, CS radar may outperform matched filtering based alternatives in some circumstances [4]. CS radar relies on the (reasonable) assumption that the delay-Doppler plane of the radar scene with a finite number of targets is sparse, meaning that what is observed is the result of a relatively few “lumped” causes that may as well be termed targets. The recoverability of CS radar is determined by its dictionary coherence, and generally a dictionary with smaller coherence coefficient (see (12)) enjoys better probability of recovery. An interesting dictionary based on time-frequency circular-shift operators is proposed in [4, 6]. Its coherence coefficient is quite close to the Welch low bound if the transmitted waveform is an Alltop sequence with a no-less-than-five prime length [2, 4, 6, 9]. It is well known that the AF determines the performance of PC radar [1, 5, 7, 8]. Is the AF still useful for CS radar performance analysis? In this paper, we show that the dictionary coherence coefficient of CS radar is identical to the second highest value of the discrete AF surface, which shares the same delay-Doppler grid decomposition as the CS dictionary. Therefore, the AF still impacts CS radar. Thus the main contribution of this paper is the formulation of the relationship between AF and the CS dictionary coherence coefficient.

The rest of this paper is organized as follows. Section 2 specifies the signal model for a moving target. The description and representation of the delay-Doppler plane are given in Section 3. The relationship between the AF and CS dictionary coherence is explored in Section 4. Section 5 provides numerical examples, and then we conclude.

Notation: Boldface uppercase and lowercase letters denote matrices and column vectors respectively; \( \mathbf{I}_k \) identifies the identity matrix with size \( k \times k \); while \( \mathbf{1}_k \) represents an all-one vector with length \( k \); \( \cdot^T \), \( \cdot^\dagger \) and \( \cdot^H \) denote transpose, conjugate, and Hermitian transpose, respectively; \( |a| \) stands for the absolute value of scalar \( a \); \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) finally, \( \text{diag}(\cdot) \) denotes the diagonal matrix formed by vector \( \cdot \), and \( \odot \) indicates the Hadamard product.

2. SIGNAL MODEL
We first consider a far-field moving target with constant range rate \( v, v \ll c \). Let \( s(t)e^{j2\pi f_o t} \) be the narrowband signal from the radar transmitter, where \( s(t) \) stands for the complex baseband transmitted pulse-shape and \( f_o \) denotes the carrier frequency. The demodulated baseband received signal from a single point target can be expressed as

\[
r(t) = \sigma s(t-\tau)e^{-j2\pi f_o \tau} + n(t)
\]

\[
\approx \sigma s(t-t_0)e^{-j2\pi f_d \tau} + n(t)
\]

(1)

where \( \sigma \) represents the complex radar scatterer coefficient, \( \bar{\sigma} = \sigma e^{-j2\pi d_0/\lambda} \), \( \tau = 2(d_0 + vt)/c \) denotes the round-trip time-varying propagation delay, \( d_0 \) measures the distance between the initial target location and the radar system, \( t_0 = 2d_0/c \), \( f_d = 2v/\lambda \) denotes the along-range Doppler frequency, \( \lambda \) is the wavelength, and \( n(t) \) is noise.

In the sampling, we assume that the size of window \( N \) is no less than the length of transmitted waveform \( s = [s_0, s_1, \cdots, s_{L-1}]^T \), where \( \|s\|_2 = 1 \); therefore, the target echo may completely or partly
Fig. 1. Cartoon of the possible composition of sampled baseband received signal with window size $N$ and waveform $s$.

fall into the sampling window as shown in Figure 1. After sampling, the received signal $r$ can be written as

$$r = \hat{\sigma} D(f_d) J_n \hat{s} + n,$$  

(2)

where $\hat{s} = [0_{2(L-N-L)/2}^T, s^T, 0_{2(L-N-L)/2}^T]^T$ stands for the extended transmitted waveform vector, $n$ is the noise vector,

$$D(f_d) = \text{diag}([1, e^{j2\pi f_d T}, \ldots, e^{j2\pi f_d (N-1)T}])$$  

(3)

denotes the $N \times N$ Doppler modulation matrix, $T$ is the sampling interval, $J_n$ is a time shift matrix and the integer $n = f(t_0)$ represents the waveform delay in the discrete time.

Define the up-shifting matrix as:

$$Z = \begin{bmatrix} I_N & 0 \\ 0_{(N-1)\times 1} & I_{N-1} \\ 0_{1\times (N-1)} & 0 \end{bmatrix}.$$  

(4)

Since the actual propagation delay of a given target can be identical to, less than or greater than the reference point – the center of the data processing window – $J_n$ is defined as

$$J_n = \begin{cases} I_N & n = 0 \\ Z^n & 0 < n \leq P \end{cases}$$  

(5)

with $P = (N + L)/2 - 1$. Clearly, $J_{-n} = J_n^T$. The operation of $J_n \hat{s}$ shifts the position of the waveform $\hat{s}$ by $n$ time units within the observation window.

The received signal $r_{M}$ for the multiple target case can be directly obtained by gathering echoes from all the points with various $(f_d, n)$,

$$r_{M} = \sum_{i} \hat{\sigma}_i D(f_{d,i}) J_{n_i} \hat{s} + n,$$  

(6)

where $\hat{\sigma}_i, D(f_{d,i})$, and $J_{n_i}$ respectively represent the radar scattering coefficient, Doppler modulation and time shift matrix for the $i$-th target.

3. RADAR SIGNAL PROCESSING: PC & CS

3.1. Pulse Compression Radar

Pulse compression obviates the tradeoff between radar probing distance (and the need for high energy) and range resolution (short pulses desired) by internal modulation of a long pulse, and has significantly contributed to the development of radar systems since the 1950s. Pulse compression estimates the radar scattering coefficient $\hat{\sigma}_i$; if it exists, of the $i$-th data processing window $W_i$ with

$$\hat{\sigma}_i = s^H W_i r = \hat{\sigma}_i s^H W_i D(f_{d,i}) J_{n_i} \hat{s} + s^H W_i n,$$  

(7)

which would be the least squares estimate of $\hat{\sigma}_i$ if there were no Doppler or delay relative to the reference point, where $W_i = [0_{L\times 1}, I_L, 0_{L\times (N-L)}]$. Repeating the procedure by increasing $i$ from 0 to $N - L$, a range profile can be obtained via sequentially stacking the estimation results $\hat{\sigma}_i$'s, and of course more detailed target information thereafter can be extracted therefrom.

The performance of a PC radar depends on its waveform AF

$$A(\tau, f_d) = \int \Re \{s(t - \tau) s^*(t) e^{j2\pi f_d t} dt,$$  

(8)

where $\Re$ denotes the support of integration. The properties of the AF underlie the tradeoffs between range and Doppler resolution and bound the performance of various PC systems: the degradation in SNR due to the mismatch between a probe and target separated by $\Delta$ is exactly $\|A(\Delta \tau, \Delta f_d)\|^2/\|A(0,0)\|^2$. And especially important here, the uncertainty (volume invariance) principle shows that the integral of $\|A(\tau, f_d)\|^2$ depends only on the signal energy: any attempt at a “thumbtack” ambiguity function in (for example) delay must necessarily produce spurious peaks elsewhere. We note that this has recently inspired matched filter-free (e.g., CS) investigation [4].

3.2. CS in the Delay-Doppler Plane

Compressed sensing is an emerging approach to solve sparse problems. Since the radar scene is generally in practice sparse, CS is a valid candidate for the joint estimation of the delays and Doppler shifts of multiple (even closely-spaced objects, or CSO) targets. To do so, the delay-Doppler plane should be divided into a fine grid, each generally with the same size $T \times f_d$. Each cell has a unique mathematical representation as well as physical explanation: for example, if a target has Doppler shift $\nu$ and time delay $\tau$ compared to the data window, its contribution can be uniquely written as $\hat{\sigma}_{\nu,\tau} D^\nu J_{\tau} s$. Now, the delay and Doppler estimation problem is recast as the search for the grid cells in which the targets lie. As the system has no knowledge of the number of targets, the received signal (2) is reformulated as

$$r = \sum_{m=1-M}^{M-1} \sum_{n=-P}^{P} \hat{\sigma}_{m,n} w_{\nu,\kappa} D^m J_{\nu} \hat{s} + n = \Phi u + n,$$  

(9)

where

- $\Phi = [\varphi_0, \varphi_1, \ldots, \varphi_{K-1}]$ denotes the time-frequency shift dictionary (TFSD) with size $N \times K$, $\varphi_k$ stands for its $k$-th atom, and $K = (2M - 1) \times (2N - 1)$. Here, $w_k$ is added to guarantee $||\varphi_k||_2 = 1$; if transmitted waveform $s$ is a constant modulus polyphase code,

$$w_{\nu} = \begin{cases} 1 & 0 \leq |\nu| \leq N/2 \\ \sqrt{L/(N - |\nu| + N/2)} & N/2 < |\nu| \leq P \end{cases}$$

due to the fact that $s^* \circ s = \frac{1}{2} I_z$.

- $u = [u_0, u_1, \ldots, u_{K-1}]$ is the sparse coefficient vector to be determined, $u_k = \hat{\sigma}_{m,n}$, and $K = (m + M - 1) \times (2P - 1) + (n + P - 1)$. If a target falls into grid $(m, n)$, $|\hat{\sigma}_{m,n}| > 0$; otherwise, $|\hat{\sigma}_{m,n}| = 0$. 

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CS poses the convex problem
\[
\min \|u\|_1 \quad \text{s.t.} \quad \|r - \Phi u\|_2 \leq \epsilon
\]  
(10)
where \(\epsilon\) bounds the noise level, and efficient solutions (e.g., [11]) are becoming available. We remark:

1. CS radar with the time-frequency circular-shift dictionary (TFCSD) has been suggested in [4]; however, since some of its atoms cannot provide a physically unique representation of delay-Doppler plane, it may not perfectly describe the target scene.

2. The size of grid element, \(T \times f_{\Delta}\), can be viewed as the delay-Doppler “resolution cell” of CS radar with perfect recovery. However, resolution cannot be indefinitely improved by increasing \(M\) and \(N\), since the dictionary coherence would vary with the grid fineness, as does the CS recoverability. Unfortunately, no analytical relationship among them has yet been found; interested readers ought to refer to [4] for more details.

4. AMBIGUITY FUNCTION IN CS RADAR

Section 3 gives the general case of a CS Radar dictionary with a physically unique and complete representation of the delay-Doppler plane; sometimes, however, people are also interested in the case with a partial delay-Doppler plane representation since targets may be concentrated on a small space. In this paragraph, we would like to investigate an extended waveform vector \(\hat{s} = [0^{T}_{(L-1) \times 1}, s^{T}, 0^{T}_{(L-1) \times 1}]^{T}\). As a result, the received signal in the delay domain \([-L+1, L-1]\) of interest is rewritten as

\[
r = \sum_{m=1-M}^{M-1} \sum_{n=1-L}^{L-1} v_{m,n} D^{m} J_{n} \hat{s} + n = \Psi v + n,
\]

(11)
where \(\Psi\) is the dictionary with \((2M-1) \times (2L-1)\) unit energy atoms \(\psi_k\) and \(v\) is the unknown sparse coefficient vector. Partial representation of delay-Doppler plane, (11), implies an assumption that no target exists outside of the \((3L-2)T/2\) length interested region; this may be true in practice. For example, if the pulse width is 20\(\mu s\), the target coverage region can be up to \(3 \times 20 \times 10^{-6} \times c/2 = 9\text{km}\), a reasonable figure for a target group.

The recoverability of an algorithm for a sparse problems is often measured by the coherence coefficient [4] of the underlying dictionary \(\Psi\), defined as

\[
\mu = \max_{i \neq j} \mu_{i,j} = \max_{i \neq j} |\langle \psi_i, \psi_j \rangle|,
\]

(12)
for \(i, j \in \{0, 1, \ldots, K-1\}\), subject to \(\|\psi_i\|_2 = 1\). The coherence coefficient quantitatively reflects the maximal similarity of the atoms. Generally, a dictionary with smaller coherence coefficient may enjoy better recoverability in probability for a given noise level. Some discussions of dictionaries with low coherence coefficients have been reported in radar [3, 4]. In the following, the relationship between the dictionary coherence and its corresponding AF is explored.

Proposition 1: Let \(\mu\) be the coherence coefficient of dictionary \(\Psi\). Then we have

\[
\mu = \max_{i,j} |A(sT, jf_{\Delta})|
\]

(13)
for \(|i| + |j| \neq 0\), where \(A(sT, jf_{\Delta})\) is the AF of waveform \(s\).

Proof. Without loss of generality, let \(\psi_i = D_{m_1} J_{n_1} \hat{s}\) and \(\psi_j = D_{m_2} J_{n_2} \hat{s}\), where \((m_1, n_1) \neq (m_2, n_2)\). Then,

\[
\mu_{i,j} = |\langle \psi_i, \psi_j \rangle| = |\langle (J_{n_1})^{H} D^m_{n_2-m_1} J_{n_2} \hat{s} \rangle|.
\]

(14)
We now enumerate different scenarios.

- If \(n_1 n_2 < 0\) and \(|n_1| + |n_2| > L - 1\), we have \(\mu_{i,j} = 0\).
- If \(n_1 n_2 \geq 0\) and \(n_2 > n_1\), or, \(n_2 \geq 0 \geq n_1\) and \(|n_1| + |n_2| \leq L - 1\), we have

\[
\mu_{i,j} = \left| \sum_{k=n_2-n_1}^{L-1} s_k^{n_2-n_1} s_k e^{j2\pi k(m_2-m_1) f_{\Delta}} \right|
\]

(15)
where \(\eta_{n_2} = e^{j2\pi (m_2-m_1) f_{\Delta} (n_1-n_2)}\).

- If \(n_1 n_2 \geq 0\) and \(n_1 \geq n_2\), or, \(n_1 \geq 0 \geq n_2\) and \(|n_1| + |n_2| \leq L - 1\), by the same reasoning, we have

\[
\mu_{i,j} = \left| A((n_1 - n_2)T, (m_2 - m_1) f_{\Delta}) \right|.
\]

(16)
The above shows that the coherence of arbitrary pairs of atoms is identical to a pair of points on the surface of \(\hat{A}(\tau, f_{\Delta})\) that has been sampled in the same way in the delay-Doppler plane.

One might also be interested in whether any discrete point of \(\hat{A}(\tau, f_{\Delta})\) can be expressed as the inner product of a pair of atoms. For any nonnegative integer \(k\), we have

\[
|A(kT, p f_{\Delta})| = \left| \sum_{l=k}^{L-1} s_{l-k} s_l^{*} e^{j2\pi p f_{\Delta}} \right|
\]

(17)
where \(l\) is an arbitrary integer to ensure \(D_{m-p+1} J_{n} \hat{s}\) and \(D_{m} J_{l} \hat{s}\) in the feasible Doppler shift area. The conclusion holds for a negative \(k\). Therefore, any point of AF can be factored as the coherence of a pair (or a group) of atoms. Therefore, Proposition 1 holds true.

Proposition 1 concisely states the relationship between AF and the dictionary coherence for the zero-padding as well as partial representation case. Proposition 1 suggests that a low second-highest value of the waveform AF on the sampling points of grid cells is desirable. This result may be helpful to waveform selection or design; the waveform with low AF sidelobes may have better performance in CS radar.

| Table 1. Targets information for the CS radar simulation |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Index | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| Delay | 3 | 2 | 1 | 5 | 8 | 9 | -5 | -9 | -7 | 0 | -8 |
| Dopp. | 0 | -1 | 0 | 6 | -8 | 2 | 5 | -5 | -3 | 3 | 0 |
| Gain | 1 | .8 | .7 | .8 | .9 | .9 | .5 | .4 | .4 | .8 | .7 |
5. NUMERICAL EXAMPLES

Due to the relationship between AF and dictionary coherence, AF may be helpful to CS radar performance analysis as well as waveform selection. In our illustrations here, a P3 code and an Alltop sequence are chosen for comparisons: their expressions are, respectively, $s_i = e^{j2\pi i L/2L}$ and $s_i = e^{j2\pi i L/\sqrt{L}}$, where $0 \leq i \leq L - 1$ and $L$ is the wavelength [2, 5], here $L = 11$. Figure 2 shows their normalized AFs; the maximum sidelobes of the Alltop sequence’s AF are less than 0.5, and the others are below 0.4, while that of P3 has more sidelobes over 0.4, and its maximal values are over 0.8. Therefore, the dictionary generated by the P3 code has higher coherence than that of the Alltop sequence; in addition, the number of atoms with significant correlation for P3 also exceeds that of Alltop.

Next, we investigate the CS information extraction ability in an additive white Gaussian noise environment via the software package “YALL1” [11]. The simulation SNR is set at 30dB (admittedly relatively high), while the error tolerance is chosen as $10^{-3}$. The delay-Doppler plane is divided into $(2L - 1) \times (2L - 1)$ grid elements of size $(T, 2\pi/(2L - 1))$, and the corresponding target locations are listed in Table 1. If the number of targets is less than seven the P3 code and the Alltop sequence have similar recoverability, e.g. Figure 3(d)–(f). When the number of targets increases further the P3 code degrades significantly, while the Alltop sequence still has acceptable results (see Figure 3(a)–(c)). However, if the target number exceeds 11, the recoverability of the Alltop sequence deteriorates, too. These phenomena may be caused by the coherence properties of the waveforms.

We observe that (1) when the number of targets is small, they may share similar recoverability; (2) Alltop sequences may have stronger targets tolerance than P3 codes in high probability; (3) CS with TFSDs have much weaker sparsity recoverability and less noise tolerance than those with random matrix dictionaries.

6. CONCLUSIONS

CS radar is a novel concept for the joint estimation of the delays and Doppler shifts of multiple targets by convex optimization rather than matched filtering. There is some thought that CS may avoid constraint by the uncertainty principle of AFs. However, we have shown that the dictionary coherence coefficient of CS radar is equal to the second largest value of the waveform AF. Since the CS recoverability relates to the dictionary coherence, AF still plays a role in the CS radar, but in a different way from in pulse compression radars. This paper does not discuss the effect of clutter, which is an important factor in radar signal processing [1, 8]. The sparsity recoverability of CS radar in clutter deserves further investigation.

7. REFERENCES