DIRECTION-OF-ARRIVAL AND SPATIAL SIGNATURE ESTIMATION IN ANTENNA ARRAYS WITH PAIRWISE SENSOR CALIBRATION

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ABSTRACT
In this paper, a new direction-of-arrival (DOA) estimation technique for partly calibrated arrays with pairwise calibrated sensors is proposed. Our approach is shown to substantially outperform several popular DOA estimation methods applicable to such partly calibrated arrays. As a byproduct, our algorithm can also blindly estimate the source spatial signatures.

Index Terms—DOA estimation, partly-calibrated arrays, sensor arrays, blind spatial signature estimation

1. INTRODUCTION
DOA estimation in partly calibrated sensor arrays is an important topic because in large arrays, a part of array geometry may be unknown [1]-[9].

One straightforward approach to DOA estimation in partly calibrated arrays is to use full array calibration algorithms [10]-[11]. However, these algorithms are usually computationally quite expensive, and such an approach does not take advantage of the known part of array geometry. Therefore, several DOA estimation techniques that are able to take into account this information have been proposed for different specific classes of partly calibrated arrays [1]-[9].

In the present paper, we propose a new DOA estimation method for sensor arrays with pairwise sensor calibration. This class of partly calibrated arrays is the same as considered in [8]. The proposed technique is based on estimating a set of candidate source spatial signatures at a given angular grid, and then obtaining the resulting DOA estimates by means of a MUSIC-type algorithm. Our approach is shown to have a substantially improved performance as compared to the rank reduction (RARE) and generalized ESPRIT estimators of [7] and [8] that are applicable to the same class of partly calibrated arrays. Moreover, as a byproduct the proposed algorithm can also blindly estimate the source spatial signatures.

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2. ARRAY SIGNAL MODEL
Consider an array of $2M$ identical omnidirectional sensors that involves two non-overlapping subarrays of $M$ sensors each. Without loss of generality, we assign the indices $1,\ldots,M$ to the sensors of the first subarray and $M+1,\ldots,2M$ to the sensors of the second subarray, respectively. Assume that the signals from $L$ ($L < M$) narrowband uncorrelated far-field (plane wave) sources impinge on the array from the unknown DOAs $\theta_1,\ldots,\theta_L$. Let the $M$ displacement vectors $\Theta = [\alpha_m, \beta_m]^T$ ($m = 1,\ldots,M$) between all the sensors of the first subarray and their corresponding pairs in the second subarray be exactly known. Here, $\alpha_m$ and $\beta_m$ are the displacements between the $m$th sensor pair along the $x$- and $y$-axes, respectively, and $(\cdot)^T$ denotes the transpose. If all the displacement vectors are equal to each other, then the array model reduces to the ESPRIT array geometry [1]. In the general case of non-identical displacement vectors, the array geometry matches to the generalized ESPRIT geometry [8].

The $t$th snapshot of the $2M \times 1$ array observation vector can be written as

$$z(t) = A(\Theta)s(t) + n(t)$$

where

$$A(\Theta) = [a(\theta_1), \ldots, a(\theta_L)]$$

is the $2M \times L$ array steering matrix, $\Theta = [\theta_1, \ldots, \theta_L]^T$ is the $L \times 1$ vector of the source DOAs, $a(\theta)$ is the $2M \times 1$ array steering vector which satisfies the property $\|a(\theta)\|^2 = 2M$, $s(t)$ is the $L \times 1$ vector of the source waveforms, $n(t)$ is the $2M \times 1$ vector of zero-mean Gaussian sensor noise with the variance $\sigma^2$ in each sensor, $(\cdot)^T$ denotes the transpose, and $\| \cdot \|$ denotes the 2-norm of a vector.

Let us introduce the $M \times M$ phase difference matrix $\Phi(\theta)$ as [8]

$$\Phi(\theta) \triangleq \text{diag}\{e^{j\phi_1}, e^{j\phi_2}, \ldots, e^{j\phi_M}\}$$

where $j \triangleq \sqrt{-1}$ and

$$\phi_m(\theta) = \alpha_m \sin \theta + \beta_m \cos \theta$$

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is the phase difference between the \( m \)th sensor pair corresponding to the direction \( \theta \). For the sake of notational brevity, let \( \Phi_l = \Phi(\theta_l) \) \((l = 1, \ldots, L)\). Using this notation, the array steering matrix \( A(\theta) \) in the considered case of pairwise calibrated sensors can be expressed as [8]

\[
A(\theta) = \begin{bmatrix}
A_1(\theta) \\
A_2(\theta)
\end{bmatrix}
\]  

(3)

where

\[
A_1(\theta) = [a_1(\theta_1), a_1(\theta_2), \ldots, a_1(\theta_L)]
\]

\[
A_2(\theta) = [\Phi_1 a_1(\theta_1), \Phi_2 a_1(\theta_2), \ldots, \Phi_L a_1(\theta_L)]
\]

are, respectively, the \( M \times 1 \) steering matrices of the first and second subarrays, and \( a_1(\theta) \) \((i = 1, 2)\) is the steering vector of the \( i \)th subarray.

The eigendecomposition of the array covariance matrix \( R \triangleq E\{z(t)z^H(t)\} \) can be written as

\[
R = E_S A_S E_S^H + E_N A_N E_N^H
\]

(4)

where \((\cdot)^H\) denotes the Hermitian transpose, and \( A_S \) and \( A_N \) are the \( L \times L \) and \((2M - L) \times (2M - L)\) diagonal matrices containing the signal- and noise-subspace eigenvalues, respectively. Correspondingly, \( E_S \) and \( E_N \) are the \( 2M \times L \) and \( 2M \times (2M - L) \) matrices that contain the signal- and noise-subspace eigenvectors, respectively.

In practice, the true array covariance matrix is unavailable and its sample estimate

\[
\hat{R} = \frac{1}{N} \sum_{t=1}^{N} z(t)z^H(t)
\]

(5)

is used, where \( N \) is the number of samples. Similar to (4), the eigendecomposition of \( \hat{R} \) can be written as

\[
\hat{R} = \hat{E}_S A_S \hat{E}_S^H + \hat{E}_N A_N \hat{E}_N^H
\]

(6)

where \( \hat{A}_S, \hat{A}_N, \hat{E}_S \) and \( \hat{E}_N \) are the sample counterparts of \( A_S, A_N, E_S \) and \( E_N \), respectively.

### 3. THE PROPOSED APPROACH

Similar to \( A \), the matrix \( E_S \) can be expressed as:

\[
E_S = \begin{bmatrix}
E_{S,1} \\
E_{S,2}
\end{bmatrix}
\]

(7)

where \( E_{S,1} \) and \( E_{S,2} \) are submatrices of \( E_S \) that correspond to the first and second subarrays, respectively.

As the signal eigenvector matrix and the array steering matrix span the same subspace, we have

\[
E_S = \Lambda T
\]

(8)

where \( T \) is some \( L \times L \) full-rank matrix. Hence, from (7) it follows that

\[
E_{S,1} = A_1 T, \quad E_{S,2} = A_2 T.
\]

(9)

From (9), we have

\[
E_{S,1} E_{S,2}^H A_2 = A_1
\]

(10)

where

\[
E_{S,2}^H = (E_{S,2}^H E_{S,2})^{-1} E_{S,2}^H.
\]

Using (10) and the relation

\[
a_2(\theta_l) = \Phi_l a_1(\theta_l)
\]

between \( a_1(\theta_l) \) and \( a_2(\theta_l) \) \((l = 1, \ldots, L)\), we have that

\[
E_{S,1} E_{S,2}^H \Phi_l a_1(\theta_l) = a_1(\theta_l), \quad l = 1, \ldots, L.
\]

(11)

Therefore, the steering vectors \( \{a_1(\theta_l)\}_{L} \) can be estimated by solving the following equation:

\[
[I_M - E_{S,1} E_{S,2}^H \Phi(\theta)]a_1(\theta) = 0
\]

(12)

subject to the constraint

\[
\|a_1(\theta)\| = \text{const}
\]

where \( 0 \) is the vector of all zeros, \( I_M \) is the \( M \times M \) identity matrix and \( [\cdot]_i \) denotes the \( i \)th entry of a vector. The latter constraint is required to avoid the trivial solution \( a_1 = 0 \).

It follows from (12) that in the finite sample case, the steering vectors can be estimated from the \( L \) main minima of the function

\[
f(\theta) = \|I_M - \hat{E}_{S,1} \hat{E}_{S,2}^H \Phi(\theta)]a_1(\theta)\|^2
\]

\[
= a_1^H(\theta) D(\theta) a_1(\theta)
\]

(13)

subject to \( \|a_1(\theta)\| = \text{const} \), where

\[
D(\theta) \triangleq I_M - \hat{E}_{S,1} \hat{E}_{S,2}^H \Phi(\theta).
\]

Therefore, the estimate of \( a_1(\theta) \) can be computed as

\[
\hat{a}_1(\theta) = M\{D(\theta)^H D(\theta)\}
\]

(14)

where the operator \( M\{\cdot\} \) returns the minor eigenvector (that is, the eigenvector corresponding to the minimal eigenvalue). Note that this eigenvector has to be rescaled so that \( |\hat{a}_1(\theta)|_1 = 1 \) based on the fact that the first sensor of the first subarray is a reference sensor.

Given the estimate (14), the steering vectors of the second subarray can be estimated as

\[
\hat{a}_2(\theta) = \Phi(\theta) \hat{a}_1(\theta)
\]

(15)
and, correspondingly, the steering vectors of the whole array can be estimated as
\[
\hat{\mathbf{a}}(\theta) = \begin{bmatrix}
\hat{a}_1(\theta) \\
\hat{a}_2(\theta)
\end{bmatrix},
\]
(16)
The next step is to estimate the DOAs by substituting (16) into the MUSIC-type function
\[
g(\theta) = \left(\hat{\mathbf{a}}^H(\theta)\tilde{\mathbf{E}}_N\tilde{\mathbf{E}}_N^H\hat{\mathbf{a}}(\theta)\right)^{-1}
\]
(17)
and obtaining them from the L largest maxima of this function.

This approach can be straightforwardly used to estimate the source DOAs in arrays with pairwise sensor calibration. However, it can be further refined by noticing that in the case of identical omnidirectional sensors, the absolute value of each component of the steering vector should be equal to one. Therefore, the proposed refinement of the estimate of \( \hat{a}_1(\theta) \) can be written as
\[
[\hat{a}_1(\theta)]_m = \frac{[\hat{a}_1(\theta)]_m}{|[\hat{a}_1(\theta)]_m|}, \quad m = 1, \ldots, M.
\]
(18)
The proposed estimation algorithm can be summarized as follows:

**Step 1:** Compute the matrices \( \tilde{\mathbf{E}}_{S,1} \) and \( \tilde{\mathbf{E}}_{S,2} \) from the eigendecomposition of \( \tilde{\mathbf{R}} \).

**Step 2:** For a chosen grid of \( \theta \), compute \( \hat{a}_1(\theta) \) using (14) and refine this estimate using (18). Compute the refined estimates of \( \hat{a}_2(\theta) \) and \( \hat{a}(\theta) \) as
\[
\hat{a}_2(\theta) = \Phi(\theta)\hat{a}_1(\theta)
\]
\[
\hat{a}(\theta) = \begin{bmatrix}
\hat{a}_1(\theta) \\
\hat{a}_2(\theta)
\end{bmatrix}
\]
respectively.

**Step 3:** Find the estimates \( \{\hat{\theta}_l\}_{l=1}^L \) of the source DOAs from the L largest maxima of
\[
p(\theta) = \left(\hat{\mathbf{a}}^H(\theta)\tilde{\mathbf{E}}_N\tilde{\mathbf{E}}_N^H\hat{\mathbf{a}}(\theta)\right)^{-1}.
\]
If necessary, also estimate the source steering vectors as \( \{\hat{\mathbf{a}}(\hat{\theta}_l)\}_{l=1}^L \).

It is worth noting that the proposed algorithm is able to estimate both the source DOAs and steering vectors (spatial signatures). The latter feature of our technique opens the avenue for its application to the receive and transmit beamforming problems, where such estimated spatial signatures can be directly used to compute beamformer weight vectors.

![Fig. 1. DOA estimation RMSEs versus SNR.](image)

**4. SIMULATIONS**

We assume an array of ten omnidirectional sensors that consists of two subarrays of \( M = 5 \) sensors each. The sensor locations (in wavelengths) of the first subarray have been chosen as
\[
(0, 0); (11.02, -6.51); (-14.03, -13.91); (6.84, 6.97); (-8.87, 11.54).
\]
The displacement vectors (in wavelengths) have been chosen as
\[
[-1.53, -0.72]^T; [-1.59, 1.39]^T; [1.78 - 2.39]^T; [0.83, 1.19]^T; [0.99, -1.48]^T.
\]

We assume two equal-power mutually uncorrelated plane-wave sources with the DOAs \( \theta_1 = 3^\circ \) and \( \theta_2 = 7^\circ \). The DOA estimation performance of the proposed technique is compared with the relevant\(^1\) deterministic Cramèr-Rao bound (CRB), the RARE algorithm of [7] and the generalized ESPRIT algorithm of [8]. Note that to boost the performance of the latter technique, we use its modified version that corresponds to equation (5.49) of [13]. \( K = 200 \) simulation runs have been used to obtain each point of our figures.

Fig. 1 shows the DOA estimation root-mean-square errors (RMSEs) of the algorithms tested versus SNR for \( N = 100 \). Fig. 2 displays the performance of spatial signature estimation of the proposed approach versus SNR for different values of \( N \). This performance has been quantified in Fig. 2 in terms of normalized RMSE
\[
\text{NRMSE} = \sqrt{\frac{1}{K} \sum_{i=1}^{K} \sum_{l=1}^{L} \frac{||\hat{\mathbf{a}}(\hat{\theta}_l) - \mathbf{a}(\theta_l)||^2}{||\mathbf{a}(\theta)||^2}}.
\]
\(^1\)Note that the results of [6] can be directly used to compute the CRB for our partly calibrated array model.
It can be observed from Fig. 1 that the proposed approach substantially outperforms the RARE and generalized ESPRIT algorithms in terms of threshold performance. Fig. 2 validates an excellent spatial signature estimation performance of our technique. The quality of spatial signature estimates can be substantially improved by increasing the sample size $N$.

5. CONCLUSIONS

A new DOA estimation technique for partly calibrated sensor arrays with pairwise sensor calibration has been proposed. Our technique offers significant performance improvements with respect to the RARE and generalized ESPRIT algorithms, and also makes it possible to blindly estimate the source spatial signatures.

6. REFERENCES


