ABSTRACT

“Performance breakdown” of maximum-likelihood (ML) direction-of-arrival (DoA) estimation is analyzed. “Performance breakdown” occurs when signal-to-noise ratio (SNR) and/or training sample volume fall below some threshold values and a ML set of DoA estimates calculated for properly detected number of sources, unavoidably contains an estimation “outlier”. In this paper, we propose a technique to “predict” (i.e. identify, recognize) the underlying scenario and an ML set of DoA estimates, as potentially containing an outlier and specify these potential outliers.

Index Terms— Parameter estimation, signal detection, maximum likelihood estimation, array signal processing.

1. INTRODUCTION AND BACKGROUND

Since [1] at least, it has been known that for scenarios with closely separated (uncorrelated) sources, the ITC-based detection of a number of such sources impinging upon an array is still highly reliable at SNR values where even the Cramér–Rao bound (CRB) suggests that accurate DoA estimation (resolution) of these sources is no longer possible. Recently [2, 3] we demonstrated that the GLRT-based detection-estimation suffers from the same “performance breakdown” as more conventional ITC-based detection, followed by ML (or ML-proxy) DoA estimation, as considered by [1]. Specifically, we showed that a covariance matrix model reconstructed with a number of sources and the associated ML DoA estimates containing an outlier can be sufficiently “likely”. In particular, a likelihood ratio (LR) for such a model reaches threshold value calculated for the LR of the true covariance matrix. At the same time, even a properly estimated model with \((m-1)\) sources does not reach this threshold, forcing the detection-estimation process to accept a model with at least one ML DoA “outlier”.

Therefore, regardless of the particular detection-estimation strategy, in closely spaced source scenarios, an “ambiguity region” where the number of sources is correct but the ML DoA set is incorrect, always exists. This means that even if in this ambiguity region we use a rigorously global ML search algorithm for the true number of sources, it does not prevent us from getting a set of DoA estimates with a severely erroneous “outlier”.

This resolution problem has been tackled via a number of methods in classical [4, 5] and more recent [6, 7] literature (to name a few), but usually through further characterization of this fundamental limit, or by the addition of ad-hoc criteria to provide separate DoA estimates in the absence of resolvable signals.

It remains quite clear that within the ML paradigm, (i.e. in the absence of any additional a priori information), we cannot replace (“cure”) an outlier by a “proper” DoA estimate, in the way we did it for a MUSIC-specific outlier in [2]. In the MUSIC case, the SNR “gap” between limits to MUSIC resolution and limits to MLE resolution was exploited to both predict the presence of an outlier and to provide an alternative DoA estimate. But in the MLE case itself, there is no such gap to exploit. Of course, there are a number of bounds and statistical measures which predict this MLE breakdown using a priori knowledge of the scenario [8, 9]. But without resorting to that non-practical approach, is there still a possibility at least to determine that the produced ML (or ML-proxy) DoA set is likely to contain an outlier and to specify the least reliable DoA estimates, even if we cannot replace them by “proper estimates”? This is a new problem that has not been addressed before, and is the subject of this paper.

In this paper, we try to provide a solution to this “ML performance breakdown prediction” problem. The central idea of our approach is, in addition to the ML (ML-proxy) set of DoA estimates, to attempt to generate a set of alternative DoA estimates, that in terms of the LR, are statistically indistinguishable from the ML (ML-proxy) DoA set. Existence of significantly different DoA sets with statistically indistinguishable LRs, are then treated as evidence of the “statistical ambiguity” of the underlying scenario. Secondly, if we are able to demonstrate that a certain DoA estimate “survives” in all of our “equally likely” DoA sets, then this estimate may be treated as “reliable” (or in contrast, “not reliable”) to augment our ML DoA estimation results with a “confidence” metric that can be incorporated into a practical treatment of ML DoA estimates.

2. RESTRICTED ML AND “EXPECTED-LIKELIHOOD” SETS OF DOA ESTIMATES

As an alternative to standard ML estimation, we may consider ML estimation within a somehow restricted admissible set, regardless whether these restrictions are true or not. Specifically, let us consider the ML DoA estimation of equally powerful sources. Note that most of the current DoA estimation techniques (MUSIC, etc.) cannot incorporate this restriction. Yet, the CRB analysis and the exhaustive search of the likelihood function may be easily adjusted to this restriction in order to investigate potential improvements in ML DoA estimation with this restriction.
Consider the scenario in Lee and Li [1] that demonstrates “ML performance breakdown” [3].

\[ M = 3, \quad T = 100, \quad m = 2 \]  \hfill (1)

Here we deal with a three-sensor uniform linear array with half-wavelength spacing, \( p_{1,2} \) and \( p_0 \) are the Gaussian source powers and the white noise power, respectively, \( \{ \theta_1^0, \theta_2^0 \} \) are the DoAs of the two sources. As demonstrated in [1, 3] reliable detection of two sources in the scenario \( \{0^\circ, 1.04^\circ\} \) by the minimum description length (MDL) ITC criterion occurs down to about \(-22\)dB. At the same time, the CRB analysis (Fig. 1) demonstrates that the predicted MSE reaches the threshold value \([\theta_2 - \theta_1]/2\) at SNR=37dB. In [3] we demonstrated that within this ambiguity range, ML DoA estimates \( \hat{\theta}_{1ML} \) and \( \hat{\theta}_{2ML} \) may be described by

\[
\hat{\theta}_{1ML} = p\hat{\theta}_{1}^0 + (1 - p)\hat{\theta}_{1ML},
\]

\[
\hat{\theta}_{2ML} = (1 - p)\hat{\theta}_{1ML} + p\hat{\theta}_{2ML},
\]

\( 0 < p \leq 1 \)  \hfill (2)

with \( p \approx 0.5 \). \( \hat{\theta}_{2ML} > \hat{\theta}_{1ML} \). Here \( \hat{\theta}_{1ML} \) and \( \hat{\theta}_{2ML} \) are the “accurate” DoA estimates in vicinity of the midpoint between the actual sources (\( \theta = 0.54 \)), with the estimated power \( \hat{p}_{1ML} = \hat{p}_{2ML} \leq 2p_1 - \alpha, \alpha \gg p_1 \). \( \hat{\theta}_{1ML} \) and \( \hat{\theta}_{2ML} \) are the “outliers” that attract the small power estimate \( \hat{p}_{out}^{1ML} = \hat{p}_{out}^{2ML} = \alpha \).

Eqn. (2) describes the typical loss of resolution by ML estimation, when one of the estimates is quite powerful and fluctuates in close vicinity of the mid-point \( \theta \) while the other estimate is a low power “outlier”. In [3] we introduced an analytical technique to accurately predict the “ambiguity range” \( \{[\hat{\theta}_{1ML}^{min}, \hat{\theta}_{2ML}^{max}]\} \). Naturally, we declare \( \hat{\theta}_{1ML} \) or \( \hat{\theta}_{2ML} \) to be an “outlier” here only because we know the true underlying scenario (1). Since in our original model no restrictions are introduced on admissible source power estimate, the fact that \( \hat{p}_{1ML}^{0} \gg \hat{p}_{ML}^{out} \) does not allow us to disregard the “weak” ML DoA estimate.

Now, let us investigate the result of “restricted” ML DoA estimation. First of all, at Fig. 1, in addition to the conventional (arbitrary) source power estimates, we introduced the CRB calculated for a priori known powers, and equal, but unknown powers. These CRBs are calculated for scenario (1) using the standard technique [10] that involves, respectively, the \( 2m \)-variate (arbitrary powers), \( m \)-variate (known powers) and \( (m+1) \)-variate (equal powers) Fisher information matrix; the white-noise power \( p_0 = 1 \) is known a priori.

This simple analysis shows the crucial dependence of CRB on the restrictions introduced on admissible signal power. Only above SNR=60dB does the predicted CRB MSE become independent of the power model, but well up to SNR=15dB(!) the predicted CRB accuracy is still below the “resolution limit” \([\theta_2 - \theta_1]/2\). In fact, if the equipower model indeed describes the true scenario (as in our case), then Fig. 1 demonstrates an “inverse” ambiguity problem. For this circumstance, the accurate ML DoA estimation is predicted by CRB for SNRs where all ITC-based detectors consistently underestimate the actual number of sources. In our case, however, we are interested in estimation breakdown prediction in circumstances where the correct number of sources (\( m = 2 \)) is still reliably detected (SNR > 22dB).

Now, we wish to demonstrate that the actual “restricted” ML DoA estimation actually follows this CRB prediction. To investigate the same ML performance (rather than that of ML-proxy algorithms), we search for the global LR maximum in two stages. First, on a finely discretized grid over the diagonal half-plane \( \theta_2 > \theta_1 \), we find the maximum LR(\( \theta_1, \theta_2 \)). We calculate the powers for the two DoAs (in the arbitrary power model following [10])

\[
[p_1, p_2] = \text{diag} \{ (S^H S)^{-1} S^H (\hat{R} - p_0 I_M) S (S^H S)^{-1} \}. \hfill (3)
\]

Here \( S = [S(\theta_1), S(\theta_2)] \in C^{M \times 2} \)

\[
S(\theta_j) = [1, \exp(c \sin \theta_j), \ldots, \exp(c(M - 1) \sin \theta_j)]^T
\]

with \( c = 2\pi d/\lambda \)

\[
\hat{R} = \frac{1}{T} \sum_{j=1}^{T} x_j x_j^H, \quad \frac{d}{\lambda} = 0.5
\]

\[
x_j \sim \mathcal{CN}(0, R_0), \quad R_0 = S_0 S_0^H + p_0 I_M
\]

\[
S_0 = [S(\theta_1^0), S(\theta_2^0)]; \quad p_0 = \text{diag} \{p_1, p_2\}; \quad p_1 = 0. \hfill (7)
\]

For the equipower model, we use

\[
\hat{p}_1 = \hat{p}_2 = \frac{1}{2M} \text{tr} (\hat{R} - p_0 I_M).
\]

For each point on a grid \( \theta_2 > \theta_1 \), say \( \{\theta_1, \theta_2, \hat{p}_1, \hat{p}_2\} \), we reconstruct the covariance matrix model

\[
\hat{R} = \hat{S} \hat{S}^H + p_0 I_M; \quad \hat{S} = [S(\theta_1), S(\theta_2)]; \quad \hat{R} = \text{diag} \{\hat{p}_1, \hat{p}_2\}
\]

and calculate the LR(\( \hat{R} \)):

\[
\text{LR}(\hat{R}) = \frac{\det \hat{R}^{-1} \hat{R} \exp \hat{M}}{\exp \text{tr} \hat{R}^{-1} \hat{R}} \leq 1
\]

which is just a normalized likelihood function for the (stochastic “unrestricted”) Gaussian model.

After the global maximum of \( \text{LR}(\hat{R}) \) over the grid is found, we use it as the initial solution to a numerical optimization routine (the Matlab function \text{fmincon}) to find the local LR maximum over the either \( \{\theta_1, \theta_2, p_1, p_2\} \) or \( \{\theta_1, \theta_2, p_1 = p_2\} \), i.e., over all four or just three parameters.

Fig. 2 shows a plot of \( \text{LR}(\theta_1, \theta_2) \) for an example realization of our scenario at 22dB SNR. The left plot is for the arbitrary power

![Fig. 1. CRB analysis for the Lee and Li scenario.](image-url)
model, while the right (zoomed) plot is for the equipower model (with the same set of training data). For the unrestricted powers, the global ML estimates are found to be

\[
\{\hat{\theta}_{1ML}, \hat{\theta}_{2ML}\} = (0.51^{\circ}, 38.28^{\circ})
\]
\[
\hat{p}_1 = 24.93dB; \quad \hat{p}_2 = -7.92dB; \quad LR = 0.961.
\]

Note that the set of ML estimates (11) is accurately described by (2), with \(\theta_{1ML}^{(0)} = 0.51^{\circ}, \Theta_{2ML} = 38.28^{\circ}\). It is important, that despite the absurdly small power of the “outlier” DoA estimate, the covariance matrix model restored with just the single powerful source in the mid-point \(\theta = 0.54^{\circ}\) in this particular case generates an LR value of just \(LR(R(\mu = 1)) = 0.843\), which is still below the threshold value \(\gamma_{FA} = 0.896\) calculated for the LR (10) of the true covariance matrix \(R_0\).

\[
\int_0^{\gamma_{FA}} \omega(\beta)d\beta = 1 - 10^{-2}; \quad \beta \sim LR(R_0)
\]

This means, that both the ITC MDL and the “expected likelihood” (EL) criterion [11] in this particular trial detect two sources, and therefore accept the two source ML DoA set (11).

If instead we assume equal powers for this same randomization, the “restricted” DoA estimates are

\[
\{\hat{\theta}_{1ML}, \hat{\theta}_{2ML}\} = (0.09^{\circ}, 0.96^{\circ})
\]
\[
\hat{p}_1 = \hat{p}_2 = 21.93dB; \quad LR = 0.927.
\]

Note that the likelihood of this “restricted” solution is closer to the likelihood of the true covariance matrix \(LR(R_0) = 0.916\) in this case. In practical applications, this specific LR value is not known, of course. Yet, the statistical invariance of the \(LR(R_0)\) p.d.f. allows us to pre-calculate an “expected likelihood” threshold in (12).

Therefore, in this particular trial we got two sets of DoA estimates (11) and (13) that generate covariance matrix models that are statistically “as likely as” the true covariance matrix \(R_0\), based on a statistical lower bound or threshold for \(LR(R_0)\). Based on comparison of the two LRs (0.961) and (0.927) with the threshold value of \(\gamma_{FA} = 0.896\) we may now declare that the underlying scenario is within the “statistical ambiguity region”, while the direct comparison of the two DoA sets (12) and (13), makes it clear that the existence of a source within the sector \(0.09^{\circ} < \theta < 0.96^{\circ}\) is very likely, while the ML DoA estimate \(\theta_2 = 38.28^{\circ}\) is likely to be an “outlier”.

Intuitively this is the expected judgement, based on the absurdly small estimated power of the second source in (11). Yet, for the original assumption on an arbitrary admissible source power, however, we could proceed no further, as we consider all source powers admissible. Incidentally, this result demonstrates that if the additional (equipower) restriction is true (as for the particular considered example), then the actual DoA estimation accuracy can indeed be significantly improved, as predicted by the CRB analysis at Fig. 1.

Note that the equipower restriction is not the only approach that may be used to get an alternative set of DoA estimates. In [12] we reported on “expected likelihood” (EL) search for two DoAs with intersource separation such that the likelihood ratio \(LR(R_0)\) for the covariance model model reaches the LR-threshold value \(\gamma_{FA} = 0.896\), in our case. Similarly to the equipower model, for the actual scenario with close (equipower) sources, this approach provides comparable to Fig. 1 accuracy. Therefore, it may be also used as instrument to produce alternative DoA sets. Alternatively, a bank of estimators could be used to provide alternative DoA estimates [13]. The conducted analysis demonstrated that by imposing additional restrictions on global ML search, one can generate a number of ML DoA estimation sets well above the “EL threshold” scenarios that...
are indeed in the “ML performance breakdown” region.

Practically though, we cannot rely upon the global ML search. Moreover, if such search is feasible as in our example, then the $LR(\theta_1, \theta_2)$ function itself reveals existence of a large continuum of solutions with statistically equal likelihood, as per left plot at Fig. 2.

Yet, based upon the revealed properties, we can propose an EL-based routine for ML “performance breakdown prediction” that does not involve the global ML search. Instead, we search for an alternative two DoA sets in vicinity of the most powerful estimated sources, as detailed in the next section.

3. ML/EL PERFORMANCE BREAKDOWN PREDICTION

1. First of all, using a conventional ITC-detection routine followed by MUSIC DoA estimation and the “ML performance breakdown prediction and cure” approach [2] we receive a set of $m$ DOA and power estimates $\{\hat{\theta}_j, \tilde{p}_j\}$ such that the LR $LR[R_m] > \gamma_{PA}$, where

$$R_m = \sum_{j=1}^{m} \tilde{p}_j S(\hat{\theta}_j) S^H(\hat{\theta}_j) + p_0 I_M, \quad p_0 = 1$$

2. To predict the “ML/EL performance breakdown” find the least contributing source to the LR achieved by sequentially excluding each source from the set of $m$ source estimates.

3. If the excluded source is the one with the minimal power (as per (11), find the most contributing to LR (typically, but not always, the most powerful) source, and in its vicinity conduct one-dimensional search (over $\Delta \theta$) for two sources at DOAs

$$\begin{cases} \theta_1 = \hat{\theta}_{max} + \Delta \theta \\ \theta_2 = \hat{\theta}_{max} - \Delta \theta \end{cases}$$

such that $LR[\Delta \hat{\theta}] > \gamma_{PA}$. (15)

Here $\hat{\theta}_{max}$ is the new DoA estimate of the most contributing source. If required, conduct such a search for all remaining DoA estimates, in descending order of their contribution to the locally optimized LR value for the $(m - 1)$ sources.

4. If the condition (15) is not met as a result of such a search, the original set of (EL) DoA estimates may be treated as reliable (i.e. outside the ML/EL “breakdown” region). Otherwise, the existence of “EL breakdown” must be declared with the excluded sources treated as the least reliable estimates.

5. If the excluded source is the one that is closely separated from the remaining one (as in (13)), conduct a limited search for the $m$-th source to maximize the overall likelihood, as per the routine in [2]. Again the existence of an alternative solution with equally high likelihood must be treated as indication of ML/EL “performance breakdown”.

4. SUMMARY AND CONCLUDING REMARKS

We have demonstrated that the onset of the “ML performance breakdown” when the ML set of DoA estimates contains an outlier while the number of sources is still properly estimated, may be reliably determined (“predicted”). Specifically, the existence of alternative set of DoA estimates with statistically indistinguishable likelihood is suggested as evidence of the “ML performance breakdown”.

For practical subspace-based DOA estimation techniques, an alternative approach is to find a set of DoA’s as suggested in (15) by noting when in the vicinity of the most powerful estimates whether the least contributing (to the LR) source has a very low power estimate. When global ML search is feasible, a pair of alternative DoA estimates may then be found using the equal source power model.

When applied to a scenario with closely separated equipower sources, these techniques dramatically improve the DoA estimation accuracy. For scenarios with arbitrary located sources with significantly different powers, these techniques allow for the prediction of “ML performance breakdown” events.

5. REFERENCES


