A TELESCOPING APPROACH TO RECURSIVE ENHANCEMENT OF NOISY IMAGES

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ABSTRACT
Images are well modeled as noncausal random fields, i.e., fields where a pixel value depends on say, its four nearest neighbors. This noncausality creates problems when processing images since it precludes the application of recursive estimators, like the Kalman filter. This paper presents a new approach that allows the application of optimal Kalman filtering to random fields, while preserving the noncausality of the image random field model. The recursions in our approach are telescoping: they initiate at the periphery (or boundary) of the random field and telescope inwards. We show how to apply the new optimal recursive Kalman filter to enhancement of noisy images.

Index Terms—Stochastic Fields, Markov processes, Recursive Estimation, Image Enhancement, Kalman filtering,

1. INTRODUCTION
Gauss-Markov random fields (GMrf’s) have been used extensively in the literature for use in image analysis, see [1] for list of references. In this paper, we study the problem of recursive enhancement of stochastic images that can be modeled by noncausal GMrf’s. For example, let \( X_{i,j}, (i,j) \in \Omega_0 = [0, N+1]^2 \), be a second order GMrf. For such fields, the minimum mean square error (MMSE) representation is such that for all \((i,j) \in [1, N]^2\),

\[
X_{i,j} = \theta_{1,0} (X_{i-1,j} + X_{i+1,j}) + \theta_{0,1} (X_{i,j-1} + X_{i,j+1}) + \theta_{1,1} (X_{i-1,j-1} + X_{i+1,j+1}) + \theta_{1,-1} (X_{i-1,j+1} + X_{i+1,j-1}) + v_{i,j},
\]

where \( E[v_{i,j}X_{k,l}] = \sigma^2 \delta_{i-k,j-l} \) and \( v_{i,j} \) is correlated noise whose covariance depends on the parameters \( \theta = [\theta_{1,0}, \theta_{0,1}, \theta_{1,1}, \theta_{1,-1}] \) and \( \sigma^2 \) [2]. Thus, each site \( X_{i,j} \) depends on its neighbors in all directions and the representation is noncausal. Much of the earlier work done in deriving recursive algorithms on random fields has assumed causal models, i.e., restricting the dependence of \( X_{i,j} \) to \( X_{i-1,j}, X_{i-1,j-1}, \) and \( X_{i,j-1} \). These authors assume causal models so they can apply standard recursive methods for causal signals, such as Kalman filtering or Rauch-Tung-Striebel smoothing [3]. However, causal models for images are not appropriate and lead to undesirable effects, such as streaking [4], [5].

Reference [4] derives recursive representations for noncausal GMrf’s under zero boundary conditions and symmetric or asymmetric Neumann boundary conditions. These recursions process the image line by line and lead recursive algorithms for GMrf’s. To allow for more general non-zero boundary conditions, we present here a new telescoping recursive representation that initiates at the boundary of the field and telescopes inwards to the center of the field. These telescoping representations can be derived for arbitrary shaped regions.

In this paper, we use the telescoping recursions to derive an optimal recursive enhancement algorithm. We first segment the original image into perceptually similar regions using a normalized cuts based image segmentation algorithm [6]. We model each segmented region as a noncausal homogeneous GMrf and estimate its parameters using a modified version of the least-squares based method developed in [7]. For estimation of noisy images, we use the Rauch-Tung-Striebel smoother [5] on the telescoping representation of each region. Experimental results on real images show that our algorithm successfully reduces the mean squared error from noisy images. Further, the use of the telescoping representation allows for processing random fields with arbitrary non-zero boundary conditions and leads to lower number of recursions. We also show that segmentation of an image into homogeneous regions further reduces the noise and leads to visually better images.

The organization of the paper is as follows. Section 2 presents the telescoping recursive representation for GMrf’s. Section 3 outlines our recursive enhancement algorithm along with experimental results. Section 4 concludes this paper.

2. TELESCOPING RECURSIVE REPRESENTATION
This Section presents the telescoping recursive representation for random fields defined on a square lattice. The extension to arbitrary shaped regions will be demonstrated in Section 3.1. Let \( X_{i,j}, (i,j) \in \Omega_0 = [0, N+1]^2 \), be a second order zero mean GMrf satisfying the noncausal correlated noise driven
The parameters \(H_k\) and \(Q_k\) in (3), given in terms of the covariance of \(Z_k\), can be calculated directly using (1). First rewrite (1) in matrix form as
\[
A\bar{X} = A_0 Z_0 + \bar{v},
\]
where \(A\) is a sparse block tridiagonal positive definite symmetric \(N^2 \times N^2\) matrix with \(N \times N\) blocks, \(\bar{X}\) and \(\bar{v}\) are \(N^2 \times 1\) column vectors formed by stacking the rows of \(X_{i,j}\) and \(v_{i,j}\), respectively, and \(A_0\) is a sparse \(N^2 \times 4(N+1)\) matrix corresponding to the interactions of the field values defined on \(\Omega_1\) with the boundary values \(Z_0\). The covariance of \(\bar{v}\) is a scaled version of \(A\) such that \(E[\bar{v} \bar{v}^T] = \sigma^2 A\).

Let \(\bar{Z} = [Z^T_1 Z^T_2 \cdots Z^T_D]^T\) be the collection of all \(Z_k\), except \(Z_0\), stacked in an \(N^2 \times 1\) column vector. The random vector \(\bar{Z}\) is a permutation of the elements in \(\bar{X}\), so we can write \(\bar{Z} = P \bar{X}\), where \(P\) is an \(N^2 \times N^2\) permutation matrix. Using the orthogonality of permutation matrices, \(P^T P = I\), multiplying (7) by \(P\), we can write (7) in terms of \(\bar{Z}\) as
\[
P A P^T \bar{Z} = P A_0 Z_0 + P \bar{v},
\]
Since \(A\) is positive definite and symmetric, so is \(P A P^T\). Thus, we can do Cholesky decomposition, so that \(P A P^T = L^T L\), where \(L\) is lower triangular. Substituting in (8) and inverting \(L^T\), we get
\[
L \bar{Z} = L^{-T} P A_0 Z_0 + \bar{w},
\]
where \(\bar{w} = L^{-T} P \bar{v}\) is white Gaussian noise since \(E[\bar{w} \bar{w}^T] = \sigma^2 I\). It can be shown that the diagonal blocks of \(L\) are lower triangular matrices of dimension \(M_{\bar{k}}^0 \times M_{\bar{k}}^0\) and the lower diagonal matrices have dimension \(M_{\bar{k}}^0 \times M_{\bar{k}-1}^0\). This leads to the derivation of the representation in (3). Further, using the sparse structure of \(A\), we can derive an iterative algorithm for computing the coefficient matrices \(H_k\) and \(Q_k\).

We have thus derived a recursive representation for \(X_{i,j}\) that initiates at the boundary of the field and telescopes inward to the center of the field. We assumed \(X_{i,j}\) is of second order, but Theorem 1 can easily be generalized to higher order fields. Although, we have presented the telescoping representation for GMrf’s on square lattices, the representation applies to fields defined on arbitrary shaped regions. We show examples of this in the next Section where we outline our algorithm for recursive image enhancement using the telescoping representation in (3).

### 3. RECURSIVE ENHANCEMENT

We assume that the noisy image \(Y_{i,j}\) has the form
\[
Y_{i,j} = X_{i,j} + \sqrt{r} n_{i,j}, (i,j) \in [1, N]^2,
\]
where \(X_{i,j}\) is the original image and \(n_{i,j}\) is mean zero white Gaussian noise with unit variance independent of \(\bar{X}\), and \(r\) is...
known. Our recursive enhancement procedure consists of the following key steps:

1) Image segmentation: Using the normalized cut based image segmentation algorithm in [6], we divide the noiseless image $X_{i,j}$ into perceptually similar regions.

2) Parameter estimation: We model each segmented region as a homogeneous GMrf satisfying (1). The unknown parameters, $\theta$ and $\sigma^2$, are estimated using a modified version of the least squares based algorithm given in [7]. Given $\theta$ and $\sigma^2$ for each segmented region, we calculate the parameters for the telescoping representation, $H_k$ and $Q_k$. Assuming Dirichlet boundary conditions, i.e., $Z_0 \sim \mathcal{N}(0, \Pi_0)$, $\Pi_0$ is estimated as the sample covariance of each region.

3) Recursive estimation: We assume that the estimator knows a priori the location of the homogeneous regions and the parameters of the telescoping representations. Since the telescoping process is a Gauss-Markov process, we apply the Rauch-Tung-Striebel [5] smoother to estimate $X_{i,j}$.

As an example, if we apply the above algorithm to an $N \times N$ image with one homogeneous region, the smoothing equations consist of a Kalman filter with $\lceil N/2 \rceil$ recursions telescoping inwards from the boundary and a smoother with $\lceil N/2 \rceil$ recursions telescoping outwards to the boundary. Thus, the total number of recursive steps is $N$, which is half the number of steps needed when compared to the recursive algorithms of [4], [5]. In the next Section, we show experimental results on applying the enhancement algorithm to real images with multiple homogeneous regions.

### 3.1. Experimental Results

As our first example, consider the original image given in Fig. 2(a). The borders in the image correspond to three different perceptually similar regions detected by the segmentation algorithm in [6]. Fig 2(b) shows the direction of the recursions in each segmented region. The recursions start at the boundary of each region, as shown by the darker shades of Fig. 2(b). The telescoping nature of the recursions is shown by the increasing intensity of pixel values towards the middle of each region. For a $300 \times 300$ image, there are about 50 recursive steps in each region. Had we not used segmentation, there would have been 150 recursive steps. Although the total number of recursions are the same, segmentation into homogeneous regions allows for distributed processing within the image.

Fig. 4(b) shows the performance of the enhancement algorithm on the noisy image in Fig. 4(a), which has a mean squared error (MSE) of 29.86dB and peak signal to noise ratio (PSNR) of 17.8dB. We see that the noise is significantly reduced in the estimated image, which has an MSE of 22.81dB and a PSNR of 24.87dB. Fig. 5 shows the performance for different levels of noise and we see that the noise has significantly been reduced in all cases (the images shown are cropped for visibility).

To show the advantage of segmenting the image into homogeneous regions, consider the image in Fig. 3(a) with four different homogeneous regions. Fig. 3(b) is the noisy image with MSE = 23.84dB and PSNR = 23.43dB. Using the enhancement method of [5], which uses a noncausal GMrf, but does not perform segmentation, the estimated image, shown in Fig. 3(c), has an MSE of 17.71dB and a PSNR of 29.60dB. If we use our enhancement algorithm, the estimated image becomes Fig. 3(d), where the MSE is reduced further to 16.81dB and the PSNR goes up to 30.46dB. Comparing the two estimated images, we see that Fig. 3(d) is visually more appealing than Fig. 3(c). This is because the segmentation of the image into homogeneous regions allows for accurate parameter estimation of different regions in the image.
4. CONCLUSION

We presented a telescoping recursive representation for Gauss-Markov random fields with non-zero boundary conditions. This new representation allowed us to derive optimal recursive estimators for noncausal random fields, while preserving the noncausal structure of fields. We applied this recursive representation to the problem of enhancement of images. The three main steps involved segmentation of the image into homogeneous regions, parameter estimation on each region, and recursive smoothing on each region. Experimental results showed that the noise was significantly reduced in the enhanced images. Further, we showed that segmentation of images into homogeneous regions amounts to superior image quality. More experimental results can be found at http://www.ece.cmu.edu/~dvats/icassp2010. Future work will involve using higher order fields and combining our enhancement algorithm with other image denoising algorithms, such as [8], [9].

5. REFERENCES


Fig. 4. Performance of recursive enhancement algorithm

(a) Noisy image with MSE = 29.86dB and PSNR = 17.8dB
(b) Estimated image with MSE = 22.81dB and PSNR = 24.87dB

Fig. 5. Left: Noisy images: MSE = 25.56dB, 28.92dB and PSNR = 22.12dB, 18.76dB (read top to bottom). Right: Estimated image: MSE = 20.21dB, 22.12dB and PSNR = 27.47dB, 25.53dB