DOUBLE SIDED CONE ARRAY FOR SPHERICAL HARMONIC ANALYSIS OF WAVEFIELDS

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ABSTRACT
This paper presents a double sided conical sensor array for common array processing applications. We show that double sided cone arrays are specially suited for extraction of spherical harmonic components of three dimensional spatial wavefields. We use a less known orthogonal relationship of spherical Bessel functions to develop a novel analysis equation for spherical harmonic extraction. As an illustration, we design and simulate a 5th order spherical harmonic decomposition array using 89 microphones to operate over a frequency band of an octave.

Index Terms— Array Signal Processing, Spherical Harmonics, Microphone Arrays, Beamforming

1. INTRODUCTION
Wavefield decomposition using spherical harmonic analysis is a useful technique employed in various array signal processing applications such as beamforming, direction of arrival estimation, soundfield recording and reproduction. In this paper, we exploit a less known radial orthogonality relationship to introduce a new conical sensor array design for wavefield decomposition.

2. PROBLEM FORMULATION
2.1. Spherical harmonic representation of wavefields
An arbitrary wavefield at a point \((r, \theta, \phi)\) expressed in the spherical coordinate system with respect to an origin, within a source free region \(\Omega\), can be written in terms of spherical harmonics [1] as

\[
S(r, \theta, \phi; k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{nm}(k) j_n(kr) Y_{nm}(\theta, \phi),
\]

where \(\alpha_{nm}(k)\) are the spherical harmonic coefficients of the wavefield, \(m\) and \(n\) \((\geq 0)\) are integers and called the order and degree respectively, \(k = 2\pi f/c\) is the wavenumber, \(f\) is frequency and \(c\) is the speed of sound, \(j_n(\cdot)\) are the spherical Bessel functions of order \(n\), and \(Y_{nm}(\cdot)\) are the spherical harmonics defined as

\[
Y_{nm}(\theta, \phi) = \frac{2n+1}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} P_{n|m|}(\cos \theta) e^{im\phi}
\]

where \(P_{n|m|}(\cdot)\) are the associated Legendre functions. Note that, the spherical harmonics \(Y_{nm}(\theta, \phi)\) are orthonormal over the unit sphere.

Equation (1) can be used to describe an arbitrary wavefield at every point in the region \(\Omega\). Knowing the spherical harmonic coefficients of a wavefield would enable to calculate the corresponding wavefield at any point in the region \(\Omega\).

2.2. Truncation
If the sources are located outside the region of interest \(\Omega\), then the wavefield becomes bounded and thus, the infinite summation in (1) can be truncated to a finite number within a given accuracy using the characteristics of the Bessel functions [2]. Let the region of interest \(\Omega\) be a sphere with radius \(R\). Then we can replace the infinite summation in (1) by \(N\) terms [3]

\[
N = [ekR/2],
\]

where \(e = \exp(1)\), with the error bounded by 4%. Another commonly used rule of thumb for truncation of (6) is \(N = [kR]\), [4]. Thus, an arbitrary soundfield within a source free spherical region of radius \(r\) could be represented by a total of \((N + 1)^2\) spherical harmonic coefficients. In other words, the dimensionality [5] of the soundfield is \((N + 1)^2\).

2.3. Analysis equation
The spherical harmonic coefficients of a wavefield can be calculated for a known wavefield over all angles on a radius \(r\) by using the orthonormal properties of the spherical harmonics as

\[
\alpha_{nm}(k) = \frac{1}{j_n(kr)} \int_{0}^{2\pi} \int_{0}^{\pi} S(r, \theta, \phi; k) Y_{nm}^*(\theta, \phi) \sin \theta d\theta d\phi
\]

(4)
provided \( j_n(kr) \neq 0 \).

Hence, in order to extract the spherical harmonic coefficients \( \alpha_{nm}(k) \), one needs to measure the wavefield. For a spherical region of interest, the most obvious geometry to measure the wavefield would be to use a spherical aperture. Spherical sensor arrays \([1, 6, 7]\) use a spherical surface to take samples of the wavefield and utilize the analysis equation (4) to estimate the coefficients.

Note that, the wavefield \( S(r, \theta, \phi; k) \) on a spherical surface of fixed radius \( r \) is mode limited to \([ekr/2]\) (3). Hence, only a limited number of coefficients can be calculated from a sphere of radius \( r \). Also the truncation of a wavefield is dependent of frequency \( k \), thus, the number of coefficients that can be determined varies with frequency. Further, the denominator of (4) approaches to zero for some values of \( kr \) and \( n \).

In this paper, we develop an alternative to the analysis equation (4), which we use to decompose a given broadband wavefield into its spherical harmonic components.

3. CONICAL APERTURE

In this section, we show how to use an array of sensors arranged on a surface of a double sided conical aperture to decompose a wavefield into spherical harmonic components.

3.1. Spherical Bessel Orthogonality

We derive below an important orthogonality relationship between the spherical Bessel functions of different orders.

**Theorem 3.1** The spherical Bessel functions \( j_n(az) \) and \( j_{n'}(az) \) are orthogonal on \(-\infty < z < \infty \) for \( z \in \mathbb{R} \) and \( a \in \mathbb{R} \):

\[
\int_{-\infty}^{\infty} j_n(az)j_{n'}(az)dz = \begin{cases} \pi/(2n+1)k & \text{if } n = n' \\ 0 & \text{otherwise} \end{cases}
\]

**Proof** The proof is given in [8].

In the next subsection we show how to use the orthogonality between spherical Bessel functions to derive an analysis equation.

3.2. Equivalent expansion

For convenience, which shall be clear later, we express (1) as

\[
S(r, \theta, \phi; k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{nm}(k) j_n(kr) P_{n|m}(\cos \theta) E_m(\phi)
\]

where

\[
E_m(\phi) \triangleq (1/\sqrt{2\pi})e^{im\phi}
\]

is the normalised exponential function and

\[
P_{n|m}(\cos \theta) \triangleq \sqrt{\frac{2n+1}{2\pi} \frac{(n-|m|)!}{(n+|m|)!}} P_{n|m}(\cos \theta)
\]

is the normalised associated Legendre function.

Note that \( E_m(\phi) \) and \( P_{n|m}(\cdot) \) are orthonormal over azimuth \( \phi \in [0, 2\pi] \), and elevation \( \theta \in [0, \pi] \), respectively.

3.3. Harmonic extraction

Consider an infinitely long double sided conical aperture with the common apex point at the origin and cone angle \( \theta_q \). Suppose there are infinite number of sensors on the surface of the conical aperture.

One can view such a conical aperture as a set of circles with varying radii forming a conical shape. Consider two such circles at \((r, \theta_q)\) and \((r, \pi - \theta_q)\) from the origin (apex)\(^1\). Using (6), we write the soundfield on the two circles as

\[
S(r, \theta_q, \phi; k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{nm}(k) j_n(kr) \times P_{n|m}(\cos \theta_q) E_m(\phi) \text{ and }
\]

\[
S(r, \pi - \theta_q, \phi; k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{nm}(k) j_n(kr) \times P_{n|m}(\cos(\pi - \theta_q)) E_m(\phi),
\]

respectively.

We have the following theorem to calculate the spherical harmonic coefficients of a wavefield using its measurements on a double cone:

**Theorem 3.2** Let \( S(r, \theta_q, \phi; k) \) be the wavefield on a double sided conical aperture where the cone axis is the z-axis, the apex is the origin, and \( \theta_q \) is the cone angle. Then the spherical harmonic coefficients of the wavefield are given by

\[
\alpha_{nm}(k) = \frac{1}{P_{n|m}(\cos \theta_q)} \frac{(2n+1)k}{\pi} \\
\times \int_{0}^{2\pi} \int_{0}^{\infty} S(r, \theta_q, \phi; k)j_n(kr) \\
\times S(r, \pi - \theta_q, \phi; k)(-1)^m j_n(-kr) E_m(\phi) dr d\phi
\]

provided \( P_{n|m}(\cos \theta_q) \neq 0 \).

**Proof** Theorem 3.2 could be proved by substituting (9) and (10) into the LHS of (11) and using the Spherical Bessel orthogonality Theorem 3.1 together with orthogonality of exponential functions (The detail of the proof is given in [8]).

Note that (11) is an alternative analysis equation to that of (4). The analysis equation (11) is based on the radial orthogonality of the spherical Bessel function evaluated on a surface

\(^1\)The notation \((r, \theta_q)\) refers to the all points given by a constant elevation \( \theta_q \) and distance \( r \) from the origin- which is a circle.
over a double sided cone whereas the standard analysis equation (4) is based on the orthogonality of associated Legendre functions evaluated over a sphere. Another distinct difference between the two analysis equations are that the conditions $P_{n|m}(\cos \theta_q) \neq 0$ and $j_n(kr) \neq 0$ should be satisfied for the existence of a solution for (11) and (4) respectively. That is there is a Legendre zero condition in (11) similar to Bessel Zero condition in (4).

A direct consequence of the Legendre zero problem is that for some $(n, m)$ values where $P_{n|m}(\cos \theta_q) \approx 0$, we can not accurately calculate harmonic coefficients. This problem can be avoided having two or more cones subtended at different angles with the same apex.

4. DISCRETE CONICAL SENSOR ARRAY

In Theorem 3.2, we show how to extract the spherical harmonic coefficients of a wavefield up to a desired order by knowing the wavefield on a double sided infinite long conical aperture. However, for practical implementation we need to truncate the cone to a finite size and to sample the continuous aperture by a finite set of discrete sensors. Thus, we approximate (11) by

$$\alpha_{nm}(k) \approx \frac{1}{P_{n|m}(\cos \theta_q)} \frac{(2n + 1)k}{\pi} \times \sum_{\ell=-L}^{L} \sum_{t=1}^{T_e} [S(r_{t\ell}, \theta_q, \phi_t; k) j_n(kr_{t\ell}) + S(r_{t\ell}, \pi - \theta_q, \phi_t; k)(-1)^m j_n(-kr_{t\ell})] E_{-m}(\phi) \Delta r_{t\ell} \Delta \phi_{t\ell}$$

(12)

where $\Delta r_{t\ell}$ are the distance between $(\ell - 1)th$ and $\ell$th circular slice of the cone aperture, $\Delta \phi_{t\ell}$ are the angular separation of sensors on the $t$th slice of the cone. The detail of the design parameters and detailed error analysis will be provided in [8].

4.1. Truncation Length

Let $N$ be the desired order of the system and let $\lambda_{mid}$ be the wavelength corresponding to the middle frequency of the desired band. Then, a good rule of thumb for truncating the cone array is given by

$$R/\lambda_{mid} = \frac{N}{\exp(1)\pi} + \frac{p}{2}$$

(13)

where $2R$ is the total length of the double cone array, $p \geq 1.5$. By increasing the value of $p$, one can reduce the errors to an arbitrary small value.

5. SIMULATION

We design a 5th order cone array operating over an octave\(^2\) for illustrative purposes. In this design, we use a cone array subtending an angle of $\theta_q = 60^\circ$ at the origin (apex), with four circles on each cone and a sensor at the origin. We set $p = 1.5$ in (13), hence $R = 1.3355 \times \lambda_{mid}$. The four circles are separated by one third of the mid band wavelength.

For simulation, we consider an audio frequency band of 3500Hz to 7000Hz. Thus, the four circles on each cone are located at 2.2cm, 4.3cm, 6.5cm, and 8.6cm from the origin. Since we are designing a 5th order array, we have 11 sensors on each circle\(^3\). Thus, there are a total of 89 sensors on the array, which includes a sensor at the origin. SNR at each sensor is set to 40dB.

We tested our design by estimating all the 36 spherical harmonic coefficients $\alpha_{nm}(k)$ for a plane wave sweeping over the entire 3D space and for all frequencies within the desired octave.

We plot normalised root mean squared error in each coefficient for $f = 3500, 5250$ and 7000 Hz in Fig. 1. We also show the error of each coefficient as bar graphs in Fig. 2. From these two error plots we notice that the performance is consistent for the three different frequencies and hence over the octave. However, there are some particular coefficients that are erroneous compared to others. For example, $\alpha_{11}$ has a 15% deviation from its theoretical value. This is due to the fact that we are using the same cone angle to find all the coefficients and at this particular angle, some of the $(n, m)$ pairs are closer to Legendre nulls. This problem could be solved by adding an additional cone at a different angle with the same apex point.

6. REFERENCES


\(^2\)The operating bandwidth could be extended by using the concepts of nested arrays [9].

\(^3\)It may be possible to have less number of sensors on smaller circles which closer to the origin, however this is not considered in this paper.
Fig. 1: Normalized error in extracted spherical harmonic coefficients for a plane wave of frequency $f = 3500, 5250$ and $7000$Hz sweeping over all angles in 3D using the designed conical array. The $(n, m)$th spherical harmonic coefficient corresponds to the coefficient number $n^2 + n + m + 1$ on the x-axis.


Fig. 2: Normalized error in harmonic coefficients $\alpha_{nm}$ for 3D double conical arrays for a plane wave sweeping over entire 3D space at (a) $f = 3500$Hz, (b) $f = 5250$Hz, (c) $f = 7000$Hz; all at SNR= 40dB.