A BLIND MONTE CARLO DETECTION-ESTIMATION METHOD FOR OPTICAL COHERENCE TOMOGRAPHY

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ABSTRACT

We consider the parametric analysis of frequency-domain optical coherence tomography (OCT) signals. A Monte Carlo (Gibbs sampler) detection-estimation method for determining the depths and reflection coefficients of tissue interfaces (reflective sites in the tissue) is proposed. Our method is blind since it estimates the instrumentation-dependent “fringe” function along with the tissue parameters. Sparsity of the detected interfaces is enforced by an impulse detector and a modified Bernoulli-Gaussian prior with a minimum distance constraint. Numerical results using synthetic and real signals demonstrate the excellent performance and fast convergence of our method.

Index Terms—Optical coherence tomography, Bayesian analysis, Gibbs sampler, Monte Carlo method, detection, estimation, Bernoulli-Gaussian model, blind deconvolution.

1. INTRODUCTION

In optical coherence tomography (OCT), the depth-dependent reflectivity of tissue are measured via an optical interferometer [1]. In the case of the retina (or other layered tissues), the OCT signal transformed to the time domain can be approximated by the convolution of a sparse impulse train (the “retina function,” abbreviated RF) with a pulse (the “fringe”). Each RF impulse corresponds to a tissue interface (reflective site in the tissue), while the fringe depends on the instrumentation and determines the imaging resolution.

Here, we propose a Monte Carlo (Gibbs sampler) detection-estimation method for determining the locations and amplitudes of the RF impulses. These correspond to the depths and reflection coefficients, respectively, of the tissue interfaces and are of clinical relevance. Our method is blind in that the fringe need not be known but is estimated along with the RF parameters. It is inspired by the Gibbs sampler based blind deconvolution method proposed for seismic signals in [2], but improves on that method due to its much faster convergence and lack of spurious impulses. These improvements are achieved by the use of a dedicated impulse detector, a sparsity-enforcing modified Bernoulli-Gaussian prior incorporating a minimum distance constraint, and a parametric model for the fringe.

Two alternative methods that have been proposed for OCT are the SMLR detection method [3, 4] and a cepstral deconvolution method [5]. The SMLR method requires the fringe to be known and may converge to a local maximum of the posterior (this does not happen with Gibbs sampler methods). The cepstral method does not perform impulse detection; furthermore, it does not consider additive measurement noise and does not compensate for modeling errors.

This paper is organized as follows. The signal model and prior distributions are discussed in Sections 2 and 3, respectively. Section 4 describes the Monte Carlo detection-estimation method. The distributions used by the Gibbs sampler are developed in Section 5. Finally, numerical results are presented in Section 6.

2. SIGNAL MODEL

Convolutions model. The sampled signal measured by a frequency-domain OCT device [1] can be modeled as (cf. [5])

\[ \hat{y}_l = |1 + R_l|^2 F_l + N_l = (1 + R_l + R_l^* + |R_l|^2) F_l + N_l, \]

where \( R_l \) and \( F_l \) are the Fourier transforms of, respectively, the RF and the fringe and \( N_l \) is measurement noise (not included in [5]). The component \( (1 \text{ caused by the interferometer’s reference branch}) \) is suppressed, along with systematic modeling errors, by a standard preprocessing procedure in which the average of the signals of many local depth scans is subtracted from \( \hat{y}_l \). Neglecting \(|R_l|^2 \ll |R_l|\) and performing an inverse discrete Fourier transform then yields the time-domain signal \( y_k = (r_k + r_k^*) \ast f_k + n_k \), where \( \ast \) denotes convolution. The component \( r_k \cdot f_k \) is effectively causal and does not overlap with the anticausal component \( r_k^* \cdot f_k \). Retaining a causal block of \( K \) samples, we obtain the convolution model

\[ x_k = r_k \ast f_k + n_k, \quad k = 0, \ldots, K-1. \]

Retina function. The RF \( r_k \) consists of irregularly spaced impulses with complex amplitudes. This can be modeled as

\[ r_k = a_k b_k, \quad k = 0, \ldots, K-1, \]

where the “impulse indicator” \( b_k \in \{0, 1\} \) indicates the impulse positions and \( a_k \in \mathbb{C} \) describes the impulse amplitudes for those \( k \) where \( b_k = 1 \) (the \( a_k \) are irrelevant elsewhere). Our goal is to detect the impulse positions (i.e., the \( k \) for which \( b_k = 1 \)) and to estimate the corresponding amplitudes \( a_k \), for an unknown fringe \( f_k \).

Denoting by \( x, r, a, f, \text{ and } n \) the length-\( K \) vectors corresponding to the sequences \( x_k, r_k, b_k, a_k, f_k, \text{ and } n_k \), respectively, the signal \( x_k = (a_k b_k) \ast f_k + n_k \) can be written as

\[ x = Fr + n = FBa + n, \quad (1) \]

where \( F \triangleq \text{toep}(f) \) is the \( K \times K \) Toeplitz matrix generated by \( f \) and \( B \triangleq \text{diag}(b) \) is the \( K \times K \) diagonal matrix with main diagonal \( b \).

Fringe. We represent the fringe \( f \) by an expansion

\[ f = \sum_{l=1}^{L} \alpha_l h_l = H \alpha, \quad (2) \]

where the length-\( K \) vectors \( h_l, l = 1, \ldots, L \) correspond to the first \( L \ll K \) Hermite functions, \( H \triangleq (h_1 \ldots h_L) \), and \( \alpha \triangleq (\alpha_1 \ldots \alpha_L)^T \). The Hermite functions cover an elliptic time-frequency region [6, 7]. Their number \( L \) and time-frequency scaling are chosen depending on the effective duration and bandwidth of the fringe.

Defining the \( K \times K \) Toeplitz matrix \( R \triangleq \text{toep}(r) \), we have \( Fr = \text{toep}(r)F = RF \), so that (1) can be written as

\[ x = RF + n = RH \alpha + n, \quad (3) \]
3. STOCHASTIC MODEL

The Bayesian methodology requires prior distributions for all quantities to be estimated or detected. In particular, we use a modified Bernoulli-Gaussian prior for the RF $r_k = a_k b_k$, as described next.

Impulse indicator. To develop the prior of the impulse indicator vector $b$, we first consider all $b_k \in \{0, 1\}$ to be independent and identically distributed (iid) Bernoulli variables. This corresponds to the auxiliary probability mass function (pmf) \( \hat{p}(b) = \prod_{k=0}^{K-1} \hat{p}(b_k) \), with \( \hat{p}(b) = p_0 \) for \( b = 0 \) and \( \hat{p}(b) = 1 - p_0 \) for \( b = 1 \). However, to exclude closely spaced impulses (which are not physically relevant and would lead to spurious impulses in the detection result), we enforce a minimum distance $d_{\text{min}}$ between any two nonzero entries $b_k = 1$ and $b_{k+1} = 1$. Let $C$ be the set of all $b$ satisfying this minimum distance condition, and let $I_C(b) = 1$ if $b \in C$ and 0 otherwise. Then, the prior pmf of $b$ is chosen (up to an irrelevant normalization) as

$$p(b) \propto I_C(b) \hat{p}(b) = \begin{cases} \hat{p}(b), & b \in C \\ 0, & \text{otherwise} \end{cases}$$

Note that the $b_k$ are no longer statistically independent.

Amplitudes. The amplitudes $a_k$ are modeled as iid zero-mean circularly complex Gaussian, i.e., the prior probability density function (pdf) of $a$ is \( p(a) = \mathcal{CN}(0, \sigma_a^2 \mathbf{I}) \), with $\sigma_a^2$ suitably chosen. Thus, \( |a_k| \) is Rayleigh-distributed, which means that amplitudes around zero have a small probability. This helps avoid small impulses (which are insignificant and typically artifacts) in the detection result.

Fringe coefficients. The fringe coefficient vector $\alpha$ in (2) is modeled as $p(\alpha) = \mathcal{CN}(0, \sigma_\alpha^2 \mathbf{I})$.

Noise. The noise is modeled as \( p(n) = \mathcal{CN}(0, \sigma_n^2 \mathbf{I}) \). Its variance $\sigma_n^2$ is treated as a random hyperparameter and estimated. The prior of $\sigma_n^2$ is chosen as an inverse gamma distribution (8):

$$p(\sigma_n^2) = T_G(\xi, \eta) = \frac{\eta^\xi}{\Gamma(\xi)} \sigma_n^{-\xi - 1} e^{-\eta/\sigma_n^2} \cdot \sigma_n^2$$

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$$u(x) \text{ is the unit step function.}$$

Joint prior and posterior. The quantities $b$, $a$, $\alpha$, and $\sigma_n^2$ to be detected or estimated are modeled as independent. Thus, their joint prior is $p(b) p(a) p(\alpha) p(\sigma_n^2)$, with $p(b)$ etc. as discussed above. Using (1), the likelihood function $p(x|b, a, \alpha, \sigma_n^2)$ is obtained by setting $n = x - F \mathbf{Ba}$ in $p(n) = \mathcal{CN}(0, \sigma_n^2 \mathbf{I})$:

$$p(x|b, a, \alpha, \sigma_n^2) = \frac{1}{\pi \sigma_n^2} \exp \left( - \frac{||x - F \mathbf{Ba}||^2}{\sigma_n^2} \right),$$

with $F =$ toep$(\mathbf{a})$ and $B = \text{diag}(b)$. For the joint posterior of $b$, $\alpha$, and $\sigma_n^2$, we then obtain $p(b, \alpha, \sigma_n^2|x) \propto p(x|b, a, \alpha, \sigma_n^2) \times p(b, a, \alpha, \sigma_n^2)$.

4. MONTE CARLO DETECTION-ESTIMATION METHOD

4.1. Detection of Impulses

Two optimal methods for detecting the impulse sequence $b = (b_k)$ are the maximum a posteriori (MAP) sequence or vector detector,

$$\hat{b}_{\text{MAP}}(x) \triangleq \arg \max_{b \in \{0, 1\}^K} p(b|x) = \arg \max_{b \in \mathbb{C}} p(b|x),$$

and the MAP component detector, also known as “maximum posterior marginal/mode (MPM) detector” (e.g., [9]),

$$\hat{b}_{\text{MPM}}(x) \triangleq \arg \max_{b_k \in \{0, 1\}} p(b_k|x), \quad k = 0, \ldots, K-1.$$
Within zero intervals, \( q(b_k = 1) \equiv 0 \), and thus the sample-based block detector will always yield the zero block. Nonzero intervals, on the other hand, have \( q(b_k = 1) > 0 \) at all positions. They can be split into smaller intervals if they are too large, or joined with neighboring intervals if they are too small. However, we often obtained good results by directly using them as blocks.

4.2. Estimation of Amplitudes, Fringe, and Noise Variance

For estimating the complex amplitudes \( a_k \) at the positions \( k \) where an impulse was detected, we ideally use the minimum mean square error (MMSE) estimator of \( a_k \) under the condition \( b_k = 1 \),

\[
\hat{a}_{k,\text{MMSE}}(x) \triangleq \mathbb{E}\{a_k | x, b_k = 1\} = \int a_k p(a_k | x, b_k = 1) \, da_k.
\]

Calculating \( p(a_k | x, b_k = 1) \) by marginalization of \( p(b, a, \alpha, \sigma_n^2 | x) \) is not feasible, so we use the sample-based approximation

\[
\hat{a}_{k,S}(x) \triangleq \frac{1}{|M_k|} \sum_{m \in M_k} a^{(m)}_k.
\]

Here, \( a^{(m)}_k \) is the amplitude of realization \( (b, a, \alpha, \sigma_n^2)^{(m)} \) at position \( k \) (i.e., the \( k \)-th entry of the \( a \) contained in \( (b, a, \alpha, \sigma_n^2)^{(m)} \)) and \( M_k \) is the set of indices \( m \) of all realizations with \( b^{(m)}_k = 1 \). We calculate \( \hat{a}_{k,S}(x) \) only at those \( k \) where an impulse was detected, i.e., where \([b^{(0)}_k(x)]_k = 1\) (this also guarantees that \( M_k \) is nonempty).

Finally, sample-based versions of the MMSE estimators of the fringe coefficient vector \( \alpha \) and the noise variance \( \sigma_n^2 \) are given by

\[
\hat{\alpha}_{S}(x) \triangleq \frac{1}{M} \sum_{m=1}^{M} \alpha^{(m)}, \quad \hat{\sigma}_n^2(x) \triangleq \frac{1}{M} \sum_{m=1}^{M} (\sigma_n^{(m)} - \mu)^2,
\]

where \( \alpha^{(m)} \) and \( (\sigma_n^{2})^{(m)} \) are, respectively, the \( \alpha \) and \( \sigma_n^2 \) of realization \( (b, a, \alpha, \sigma_n^2)^{(m)} \).

5. GIBBS SAMPLER

For drawing a sample \( S = \{ (b, a, \alpha, \sigma_n^2)^{(m)} \}_{m=1}^{M} \) from the joint posterior \( p(b, a, \alpha, \sigma_n^2 | x) \) we use the Gibbs sampler [10]. Assuming a generic multivariate distribution \( p(z) \), the Gibbs sampler draws a sample \( \{ z^{(m)} \}_{m=1}^{M} \) from \( p(z) \) in a recursive fashion.

A new realization \( z^{(m+1)} \) is obtained from the current realization \( z^{(m)} \) as follows. Each entry \( z^{(m+1)}_k \) is drawn from the univariate distribution \( p(z_k | z_{-k}) \), where \( z_{-k} \) denotes \( z \) without entry \( z_k \) and the values used in the condition \( z_{-k} \) are the most recent ones (i.e., in general, some of them have already been updated). The stationary distribution of the resulting Markov chain \( z^{(m)} \), \( m = 0, 1, \ldots \) converges to \( p(z) \) as \( M \to \infty \) [10]. Thus, the sample \( \{ z^{(m)} \}_{m=1}^{M} \) will represent \( p(z) \) if it is large enough and if an initial “burn-in period” (which is affected by the initialization of \( z^{(0)} \)) is removed.

The univariate distributions \( p(z_k | z_{-k}) \) can be derived from the likelihood function (in our case (5)) and the priors (see Section 3). Because of space limitations, we will present them without proof.

Amplitudes. Using the fact that \( p(a_k) \) is a conjugate prior [11], we obtain \( p(a_k | x, b, a_{-k}, \alpha, \sigma_n^2) = \mathcal{CN}(\mu, \sigma^2) \), with

\[
\mu = \frac{\sigma_n^2 \mathbb{E}(x - FB_{-k}a_{-k})}{\sigma_n^2}, \quad \sigma^2 = \frac{(b_k^2 \|f_k\|^2 + 1)}{\sigma_n^2}.
\]

Here, \( f_k \) denotes the \( k \)-th column of \( F \) and \( B_{-k} \) denotes \( B \) without the \( k \)-th column.

Impulse indicator. For \( b_k \), we find

\[
p(b_k | x, b_{-k}, a_{-k}, \alpha, \sigma_n^2) = C \frac{e^{-(x - FB_{-k}a_{-k})^2 / 2 \sigma_n^2}}{\sigma_n^2},
\]

with \( \mu \) and \( \sigma^2 \) as defined in (9). The constant \( C \) is easily calculated since \( b_k \in \{0, 1\} \). However, using (10), many Gibbs sampler iterations are needed to obtain changes in \( b \). This is because due to the minimum distance condition, \( b_k = 1 \) is almost determined by its neighborhood. (A similar phenomenon was observed in [12, 13].)

We will thus use a modified distribution whose condition ignores not just \( b_k \) but all \( b_{kn} \) within a neighborhood of length \( d_{\min} \) of position \( k \), and the dimension at which we sample. This neighborhood is defined by \( K \triangleq [k, d_{\min}] \) with \( d_{\min} \equiv \min\{(k + d_{\min} - 1, K) \} \) (we assume that the \( b_{kn} \) for \( k' < k \) have already been sampled). We then use the Gibbs sampler with \( p(b_k | x, b_{-k}, a_{-k}, \alpha, \sigma_n^2) \) replaced by \( p(b_k | x, b_{-k}, a_{-k}, \alpha, \sigma_n^2) \), where, e.g., \( b_{-k} \) denotes \( b \) without the entries indexed by \( K \). This latter pmf can be calculated as \( p(b_k | x, b_{-k}, a_{-k}, \alpha, \sigma_n^2) = \mathbb{P}_{\mathcal{B}_{-k}} p(b_k | x, b_{-k}, a_{-k}, \alpha, \sigma_n^2) \), where \( \mathbb{P}_{\mathcal{B}_{-k}} \triangleq (b_k b_{k+1} \cdots b_{k+b_{kn}})^2 \) comprises all \( b_k \) indexed by \( K \).

Note that there are only \( d_{\min} + 1 \) different \( \beta \) because in \( K \) there can be at most one impulse. One can show that \( p(b_k | x, b_{-k}, a_{-k}, \alpha, \sigma_n^2) \) is still given by expression (10), however with

\[
\mu = \frac{\sigma_n^2 \mathbb{E}(x - FB_{-k}a_{-k})}{\sigma_n^2}, \quad \sigma^2 = \frac{(H_i^H R_i H_i + \frac{1}{\sigma_n^2})^{-1}}{\sigma_n^2}.
\]

Here, \( R_i \) consists of the columns of \( R_i \) indexed by \( K, F_{-k} \); \( \Sigma \) is \( \Sigma = \frac{(H_i^H R_i H_i + \frac{1}{\sigma_n^2})^{-1}}{\sigma_n^2} \).

The time-shift ambiguity inherent to the convolution model is resolved as discussed in [2].

Noise variance. Since \( \sigma_n^2 \) in (4) is a conjugate prior, we obtain

\[
\sigma_n^2 = \frac{\mathbb{E}(x - FB_{-k}a_{-k})^2}{\sigma_n^2} - \frac{\mathbb{E}(x - FB_{-k}a_{-k})^2}{\sigma_n^2}.
\]

6. NUMERICAL RESULTS

We next demonstrate the performance of our method and compare it with that of the Gibbs sampler based blind deconvolution method of [2], hereafter referred to as “reference method” (RM).

Synthetic signals. We simulated 100 RFS \( r_k = a b_k \) and corresponding noisy time-domain signals \( x_k = r_k + f_k \), using a fixed fringe with Gaussian shape and height 1 (see the right-hand plots in Fig. 1(b), (c)), a fixed noise variance \( \sigma_n^2 \), and pmf/pdf parameters \( p_0 = 0.96, d_{\min} = 12 \), and \( \sigma_n^2 = 7.74 \times 10^{-5} \). We chose \( \sigma_n^2 \) to obtain an SNR \( \| F_i \| / \| n \| \) of 16 dB. Both methods used the true \( p_0, d_{\min}, \sigma_n^2 \) and, in (4), \( \eta = 0.5 \) and the value of \( \xi \) for which the mean \( \eta / (\xi - 1) \) equals the true \( \sigma_n^2 \). We used \( L = 7 \) Hermite functions in our method and a fringe length of 33 in the RM.

Fig. 1 shows a typical example. While the results of both methods are seen to be satisfactory (with some amplitude estimation errors and spurious impulses in the RM result), our method requires only three iterations to achieve a better result than does the RM with 4000 iterations (the number of iterations used in [2]). Since one iteration of our method is similarly complex as four iterations of the RM, our three iterations correspond to about 12 RM iterations. However, the RM results after 12 iterations (not shown) are still extremely poor.
Fig. 1. Results for a synthetic signal: (a) Signal, (b) RF estimate (left) and fringe estimate (right) obtained with the proposed method after three iterations, (c) RF estimate (left) and fringe estimate (right) obtained with the reference method after 4000 iterations. The circles indicate the true impulse positions and amplitudes. The dashed lines in the right-hand plots (almost coinciding with the solid lines) indicate the true fringe function. Real parts are shown throughout.

Fig. 2 shows the empirical normalized mean-square error (NMSE) of RF estimation versus the number of iterations. The empirical NMSE is the average (over 100 realizations) of $\|\hat{r} - r\|^2$ normalized by the average of $\|r\|^2$. In our method, we resynthesized $\hat{r}$ from the detected impulses and estimated amplitudes. (Note that the RM directly yields $r$.) The number of iterations equals the total length of the Markov chain. Out of each chain, the last 25% of the iterations were used to constitute the sample. It is seen that the RM converges very slowly, whereas our method achieves an NMSE below $-20$ dB after only three iterations. We note that an improvement of the RM resulting in faster convergence was proposed very recently [13].

**Real OCT signal.** We applied the two methods to a real OCT signal measured by a frequency-domain OCT device (frequency range 320.7–425.7 THz) scanning a human retina. The parameters (chosen for best performance) were $p_0 = 0.85$, $d_{\text{min}} = 7$, $\sigma_a^2 = 3.91 \cdot 10^{-5}$, $\text{SNR} = 26$ dB, and $L = 30$ for our method and $p_0 = 0.973$, $\sigma_a^2 = 8.79 \cdot 10^{-5}$, $\text{SNR} = 16$ dB, and fringe length $33$ for the RM. Fig. 3(a) shows a segment of the time-domain signal corresponding to a length of 530 μm. From Fig. 3(b),(c), it is seen that the RF estimate obtained with our method after 20 iterations is quite different, whereas that obtained with the RM after 4000 iterations is not. The two fringe estimates are quite different; a validation of these estimates based on fringe measurements will be part of our future work.

**7. CONCLUSION**

We proposed a Gibbs sampler method for inferring the depths and reflection coefficients of tissue interfaces from signals measured by frequency-domain OCT instrumentation. Knowledge of the instrumentation-dependent fringe function is not required. Our method enforces sparsity of the tissue interfaces by using an impulse detector and a modified Bernoulli-Gaussian prior with a minimum distance constraint. Numerical results demonstrated significant performance gains compared to the blind deconvolution method proposed in [2].

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**9. REFERENCES**


