ONE-DIMENSIONAL MODE ALGORITHM FOR TWO-DIMENSIONAL FREQUENCY ESTIMATION

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Abstract
This paper describes how the computationally efficient one-dimensional MODE (1D-MODE) algorithm can be used to estimate the frequencies of two-dimensional complex sinusoids. We show that the 1D-MODE algorithm is computationally more efficient than the asymptotically statistically efficient 2D-MODE algorithm, especially when the numbers of spatial measurements are large. We find that the 1D-MODE algorithm is asymptotically statistically efficient for high signal-to-noise ratio. We also show that although 1D-MODE is no longer statistically efficient when the number of temporal snapshots is large, the performance of 1D-MODE can still be very close to that of the 2D-MODE under mild conditions. Numerical examples comparing the performances of the 1D-MODE and 2D-MODE algorithms are also presented.

1. INTRODUCTION
In [1], we presented a two-dimensional MODE (2D-MODE) algorithm for estimating 2-D frequencies. There are many applications for 2-D frequency estimation, which include angle-of-arrival estimation with a 2-D sensor array and synthetic aperture radar imaging [1]. Compared with the exact maximum likelihood estimator, the 2D-MODE algorithm avoids the multidimensional search over the parameter space [2]. Yet 2D-MODE has been shown to be statistically efficient under either the assumption that the number of temporal snapshots is large or the signal-to-noise ratio (SNR) is high.

The purpose of this paper is to describe how the computationally efficient one-dimensional MODE (1D-MODE) algorithm [9] can be used to estimate the frequencies of 2-D complex sinusoids. We show that the 1D-MODE algorithm is computationally more efficient than the 2D-MODE, especially when the numbers of spatial measurements are large. We also find the 1D-MODE algorithm is statistically efficient for high signal-to-noise ratio (SNR). Even though the 1D-MODE algorithm is no longer statistically efficient when the number of temporal snapshots is large, its performance can still be very close to that of the 2D-MODE under mild conditions. Numerical examples comparing the performances of the 1D-MODE and 2D-MODE algorithms are included in this paper.

2. PROBLEM FORMULATION
Consider the following model of 2-D complex sinusoids in additive noise:

\[
y_{m,n}(t_n) = \sum_{k=1}^{K} \sum_{k=1}^{K} \alpha_{k,k}(t_n) e^{j(\omega_k m + \mu_k n)} + e_{m,n}(t_n),
\]

where \( m = 1, 2, \ldots, M, \bar{m} = 1, 2, \ldots, \bar{M} \), and \( n = 1, 2, \ldots, N \). We refer to \( M (M > K) \) and \( \bar{M} (\bar{M} > \bar{K}) \) as the numbers of spatial measurements, and to \( N \) as the number of temporal snapshots. The additive noise \( e_{m,n}(t_n) \) is assumed to be a complex Gaussian random process with zero-mean and

\[
E\{e_{m,n}(t_n)e_{m,n}^*(t_n)\} = \sigma^2 \delta_{n_1,n_2},
\]

where \((\cdot)^*\) denotes the complex conjugate and \( \delta_{n_1,n_2} \) denotes the Kronecker delta. The \( e_{m,n}(t_n), \bar{m} = 1, 2, \ldots, M, \bar{m} = 1, 2, \ldots, \bar{M} \), are also assumed to be independent of each other and the complex sinusoids. The complex amplitudes \( \alpha_{k,k}(t_n), k = 1, 2, \ldots, K, \bar{k} = 1, 2, \ldots, \bar{K} \), may be modeled either as the stochastic (or unconditional) signal model or as the deterministic (or conditional) signal model [4, 5].

Let \( Y(t_n) \) and \( E(t_n) \) be \( M \times \bar{M} \) matrices whose \((m, \bar{m})\)th elements, respectively, are \( y_{m,n}(t_n) \) and \( e_{m,n}(t_n) \). Define \( X(t_n) \) to be a \( K \times \bar{K} \) matrix whose \((k, \bar{k})\)th element is \( \alpha_{k,k}(t_n) \). Let

\[
A = \begin{bmatrix} a(\omega_1) & \cdots & a(\omega_K) \end{bmatrix},
\]

\[
a(\omega_k) = \begin{bmatrix} e^{j\omega_k} & \cdots & e^{jM\omega_k} \end{bmatrix}^T,
\]

\[
B = \begin{bmatrix} b(\mu_1) & \cdots & b(\mu_{\bar{K}}) \end{bmatrix},
\]

and

\[
b(\mu_k) = \begin{bmatrix} e^{j\mu_k} & \cdots & e^{j\bar{M}\mu_k} \end{bmatrix}^T,
\]

where \( k = 1, 2, \ldots, K, \bar{k} = 1, 2, \ldots, \bar{K} \), and \((\cdot)^T\) denotes the transpose. Then \( Y(t_n) \) can be written as

\[
Y(t_n) = AX(t_n)B^T + E(t_n).
\]

The problem of interest herein is to estimate \( \omega_1, \omega_2, \ldots, \omega_K \) and \( \mu_1, \mu_2, \ldots, \mu_{\bar{K}} \) from \( Y(t_n), n = 1, 2, \ldots, N \).

3. 2-D FREQUENCIES ESTIMATES WITH 1D-MODE
First consider using 1D-MODE to estimate \( \omega = [\omega_1, \omega_2, \ldots, \omega_K]^T \).

Let

\[
\hat{\omega} = \frac{1}{N} \sum_{n=1}^{N} Y(t_n)Y^H(t_n),
\]

where \((\cdot)^H\) denotes the complex conjugate transpose and \( \hat{\omega} \) is the estimate of the following spatial covariance matrix:

\[
R_{\omega} = E\{Y(t_n)Y^H(t_n)\} = AP\omega A^H + \bar{M}\sigma^2 I,
\]
where $E\{\cdot\}$ denotes the expectation and
\[
P_\omega = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E \left\{ X(t_n)B^TB^*X^H(t_n) \right\},
\] (10)
with $(\cdot)^*$ denoting the complex conjugate. Note that we use (10) to accommodate both the deterministic and stochastic signal models. The 1D-MODE algorithm [5], or, in a related form, WSF [6], can be applied to $\hat{R}_{\omega}$ to obtain the estimate of $\omega$, as shown below.

Let the columns in $\hat{E}_{\omega}$ be the signal subspace eigenvectors of $\hat{R}_{\omega}$ that correspond to the $\hat{K}$ largest eigenvalues of $\hat{R}_{\omega}$, where
\[
\hat{K} = \min(MN, \text{rank}(P_{\omega})),
\] (11)
We assume that $\hat{K}$ is known. If $\hat{K}$ is unknown, it can be estimated from the data as described, for example, in [7]. Further, let $\hat{A}_\omega$ be a diagonal matrix with diagonal elements $\hat{A}_\omega = \hat{A}_\omega - \bar{M} \sigma^2 I$,
\[
(\hat{A}_\omega)^2 = \frac{1}{M-M-K} \sum_{i=K+1}^{M} \hat{A}_i
\] (12)
with $\hat{A}_\omega$ denoting the identity matrix and
\[
\sigma^2 = \frac{1}{M-M-K} \left[ \text{tr} (\hat{R}_{\omega}) - \sum_{i=1}^{K+1} \hat{A}_i \right].
\] (13)

The value of $\sigma^2$ is obtained by minimizing the following criterion:
\[
f(\omega) = \text{tr} \left[ (P_\omega^T(\omega) \hat{E}_{\omega}^2 \hat{A}_\omega^{-1} (\hat{E}_{\omega})^H) \right],
\] (14)
where, for some matrix $Z$, the symbol $P_Z^T$ stands for the orthogonal projector to the null space of $Z$. To compute the estimate of $\omega$ without searching over the parameter space, the projector $P_Z^T$ above must be reparameterized in terms of the coefficients of the so-called “linear predictor” polynomial [1, 3, 5]. We remark that $\hat{\omega}$ is a consistent estimate of $\omega$ for either large $N$ or high SNR [5, 6].

Let
\[
\hat{R}_\mu = \frac{1}{N} \sum_{n=1}^{N} X(t_n)^T(t_n) Y^*(t_n)
\] (15)
be the estimate of the following spatial covariance matrix:
\[
R_\mu = E \left\{ Y^T(t_n) Y^*(t_n) \right\} = BP\mu B^H + M \sigma^2 I,
\] (16)
where
\[
P_\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E \left\{ X^T(t_n)A^TA^*X^*(t_n) \right\},
\] (17)
Similarly, the 1D-MODE algorithm can be applied to $\hat{R}_\mu$ to obtain the estimate $\hat{\mu}$ of $\mu$.

We remark that the amount of computations required by the 2D-MODE algorithm is $O(M^2M^2N)$ and that required by 1D-MODE is $O(MMM + M)$. Since the 2D-MODE algorithm requires the computation and eigen decomposition of an $M^2M^2$ matrix $\hat{R} [1]$, where
\[
\hat{R} = \frac{1}{N} \sum_{n=1}^{N} \text{vec} \left[ Y^T(t_n) \right] \text{vec}^H \left[ Y^T(t_n) \right],
\] (18)
with vec$(\cdot)$ denoting stacking all columns of a matrix into a single column vector, while both $\hat{R}_\omega$ and $\hat{R}_\mu$ can be formed from only the diagonal blocks of $\hat{R}$. Thus for large $M$ and $M$, 1D-MODE requires much less computations than 2D-MODE.

4. STATISTICAL PERFORMANCE ANALYSIS

4.1. The Case of High SNR.

In the case of high SNR, i.e., $\sigma \ll 1$ (whereas $P_\omega$ and $P_\mu$ are $O(1)$), the asymptotic covariance matrices of the estimate $\hat{\omega}$ of $\omega$ and $\hat{\mu}$ of $\mu$ are respectively given by
\[
E \left\{ (\hat{\omega} - \omega)(\omega - \omega)^T \right\} = \frac{\sigma^2}{2N} \left[ \text{Re} (H_{\omega} \circ \hat{P}_{\omega}) \right]^{-1},
\] (19)
\[
E \left\{ (\hat{\mu} - \mu)(\mu - \mu)^T \right\} = \frac{\sigma^2}{2N} \left[ \text{Re} (H_{\mu} \circ \hat{P}_{\mu}) \right]^{-1},
\] (20)
where $\circ$ denotes the Hadamard-Schur matrix product (i.e., elementwise multiplication),
\[
\hat{P}_\omega = \frac{1}{N} \sum_{n=1}^{N} X(t_n)B^TB^*X^H(t_n),
\] (21)
\[
\hat{P}_\mu = \frac{1}{N} \sum_{n=1}^{N} X^T(t_n)A^TA^*X^*(t_n),
\] (22)
\[
H_{\omega} = D\hat{\omega}^\dagger D_{\omega},
\] (23)
with the $k$th column of $D_{\omega}$ being $\partial a(\omega_k)/\partial \omega_k$, and
\[
H_{\mu} = D\hat{\mu}^\dagger D_{\mu},
\] (24)
with the $k$th column of $D_{\mu}$ being $\partial b(\mu_k)/\partial \mu_k$.

It has also been shown in [1] that the asymptotic (for SNR $\gg 1$) covariance matrices of the estimate of $\omega$ and $\mu$ obtained with 2D-MODE are equal to the corresponding deterministic Cramer-Rao bound (CRB) given by
\[
\left[ (\text{CRB}_\omega^{-1})^{-1} \right]_{i,j} = \left( \frac{2N}{\sigma^2} \right) \text{Re} \left\{ \left[ \left( A_{i}^H P_{\omega} A_{i}^H \right) \otimes \left( B_i^H B_i \right) \right] \hat{S} \right\},
\] (25)
\[
\left[ (\text{CRB}_\mu^{-1})^{-1} \right]_{i,j} = \left( \frac{2N}{\sigma^2} \right) \text{Re} \left\{ \left[ \left( A^H A \right) \otimes \left( B_i^H B_i \right) \right] \hat{S} \right\},
\] (26)
where $\otimes$ denotes the Kronecker product, $A_{i} = \partial A(\omega_i)/\partial \omega_i$, $B_i = \partial B(\mu_i)/\partial \mu_i$, and $\hat{S} = \frac{1}{N} \sum_{n=1}^{N} s(t_n)s^H(t_n)$ with $s(t_n) = \text{vec}[X^T(t_n)]$.

From a straightforward computation of Equations (19) and (20), we have
\[
E \left\{ (\hat{\omega} - \omega)(\omega - \omega)^T \right\} = \text{CRB}_\omega^{\hat{\omega}},
\] (27)
\[
E \left\{ (\hat{\mu} - \mu)(\mu - \mu)^T \right\} = \text{CRB}_\mu^{\hat{\mu}}.
\] (28)
Thus the estimates \( \hat{\omega} \) and \( \hat{\mu} \) obtained with the 1D-MODE asymptotically (for high SNR) achieve the CR-bounds in (25) and (26), respectively, which means that the 1D-MODE is an asymptotically (for SNR \( \gg 1 \)) statistically efficient estimator for estimating the 2-D frequencies.

4.2. The Case of Large \( N \)

Similar to the case of high SNR, the asymptotic covariance matrices of \( \hat{\omega} \) and \( \hat{\mu} \) in the case of large \( N \) are respectively given by

\[
E \left\{ (\hat{\omega} - \omega)(\hat{\omega} - \omega)^T \right\} = \frac{\sigma^2}{2N} \left[ \text{Re} \left( H_\omega \odot V_\omega^T \right) \right]^{-1},
\]

(29)

and

\[
E \left\{ (\hat{\mu} - \mu)(\hat{\mu} - \mu)^T \right\} = \frac{\sigma^2}{2N} \left[ \text{Re} \left( H_\mu \odot V_\mu^T \right) \right]^{-1},
\]

(30)

where

\[
V_\omega = P_\omega A^H R_\omega^{-1} A P_\omega,
\]

(31)

and

\[
V_\mu = P_\mu B^H R_\mu^{-1} B P_\mu.
\]

(32)

It has been shown in [1] that the large-sample covariance matrices of the estimates of \( \omega \) and \( \mu \) obtained with 2D-MODE are equal to the corresponding stochastic CRBs given by

\[
[(\text{CRB}_\omega^s)^{-1}]_{ij} = \frac{(2N/\sigma^2)\text{Re}}{\text{tr} \left\{ \left[ (A_i^H P_A^+ A_i) \otimes (B_i^H B_i) \right] S (A^H \otimes B^H) R^{-1} (A \otimes B) S \right\}},
\]

(33)

and

\[
[(\text{CRB}_\mu^s)^{-1}]_{ij} = \frac{(2N/\sigma^2)\text{Re}}{\text{tr} \left\{ \left[ (A^H A) \otimes (B_i^H P_B^+ B_i) \right] S (A^H \otimes B^H) R^{-1} (A \otimes B) S \right\}},
\]

(34)

respectively, where

\[
S = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} E \left\{ s(t_n) s^H(t_n) \right\},
\]

(35)

and

\[
R = E \left\{ \text{vec} \left[ Y^T(t_n) \right] \text{vec}^H \left[ Y^T(t_n) \right] \right\}.
\]

(36)

In this case, the 1D-MODE is no longer an asymptotically statistically efficient estimator for estimating the 2-D frequencies. According to the general theory of the CR-bounds, we have

\[
E \left\{ (\hat{\omega} - \omega)(\hat{\omega} - \omega)^T \right\} \geq \text{CRB}_\omega^0,
\]

(37)

and

\[
E \left\{ (\hat{\mu} - \mu)(\hat{\mu} - \mu)^T \right\} \geq \text{CRB}_\mu^0.
\]

(38)

The numerical examples given in the following section show that the larger the \( M \) (\( M \)) or the higher the SNR, the smaller the difference between \( E \left\{ (\hat{\omega} - \omega)(\hat{\omega} - \omega)^T \right\} \) and \( \text{CRB}_\omega^0 \) (\( \text{CRB}_\mu^0 \)).

4.3. Further Comments

We remark that when \( X \) in (7) is a diagonal matrix, \( y_m, m(t_n) \) can be modeled with the following data model [1]:

\[
y_m, m(t_n) = \sum_{k=1}^{K} \alpha_k(t_n) e^{i(\omega_0 m + \theta_0 + m \phi)} + c_m, m(t_n).
\]

(39)

For this case, the 1D-MODE approach can again be used to estimate the 2-D frequencies. Yet it can be shown that serious performance degradation can occur when any of the 1-D approaches (including 1D-MODE) is used with (39), which makes our results even more unexpected. An intuitive explanation is that when \( X \) is a diagonal matrix, 1-D processing does not exploit all of the information available and hence lacks the statistical efficiency. When \( X \) is a full matrix, which is the case we assume, there is no structural information that is missed by 1-D processing and hence there is no performance degradation under mild conditions.

We also remark that the 1D-MODE approach can be applied to data with non-Gaussian noise without any modification. Its asymptotic covariance matrices will be the same. The CRB matrix for the non-Gaussian case will be different from the one for the Gaussian case, but the CRB matrix computed under the Gaussian assumption remains the lower bound for a large class of estimators whose asymptotic covariance matrices do not depend on the data distribution (for instance all estimators based on second-order statistics).

5. NUMERICAL RESULTS

Figure 1. Direction-of-arrival estimation with a 2-D array.

We present below an example showing the performance of the 1D-MODE algorithm. The empirical results of the algo-
and \( \{ \mu_k \} \) in Equation (1) by:

\[
\omega_k = \frac{2 \pi \delta_1}{\lambda_0} \sin \theta_k \sin \phi_k, \quad k = 1, \tag{40}
\]

and

\[
\mu_k = \frac{2 \pi \delta_2}{\lambda_0} \cos \theta_k, \quad k = 1, 2, \tag{41}
\]

where \( \lambda_0 \) denotes the wavelength of the incident signals. In our examples, the SNR of the reflected signal is assumed to be 3 dB less than that of the direct signal. Further, the correlation coefficient between the direct and the reflected signals is 0.99. The spacings \( \delta_1 \) and \( \delta_2 \) between two adjacent sensors in the array are assumed equal to a half wavelength. The asymptotic variances of the estimates \( \phi_k \) and \( \theta_k \) are readily obtained from the asymptotic variances of the estimates of \( \omega_k \) and \( \mu_k \) given in Section 4 and the Equations (40) and (41) relating \( \phi_k \) and \( \theta_k \) to \( \omega_k \) and \( \mu_k \).

Figure 2 shows the root-mean-squared errors (RMSEs) of the angle estimates and the corresponding asymptotic (for large \( N \)) statistical performances of the 1D-MODE and 2D-MODE algorithms as a function of the SNR of the direct signal when \( \phi = 45^\circ \), \( \theta = 85^\circ \), and \( N = 500 \). As expected, as the SNR increases, the performance of 1D-MODE approaches that of the 2D-MODE, whose asymptotic statistical performance is also equal to the corresponding CR-bound. The performance of 2D-MODE is slightly better than that of 1D-MODE only when the SNR is very small. Yet the amount of computations needed by 1D-MODE was only about 7% of that required by 2D-MODE in our simulations. Further, as \( M \) and \( M \) increase, more computational savings can be achieved by using 1D-MODE.

REFERENCES


