SIGNAL DE-NOISING USING THE WAVELET TRANSFORM AND
REGULARIZATION

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ABSTRACT

This paper 1 presents a new signal de-noising algorithm using wavelets. We have developed a filtering scheme in the
wavelet domain, that involves selective smoothing at each scale of the time-frequency plot. The amount of smoothing
is controlled by regularizing factors, and gradient-based switches are used to avoid distortion of signal features. The
algorithm is seen to compare favorably to that of Mallat et al [5, 6], as it is able to recover both the smooth portions as
well as Brownian texture in the input, from the noisy signal.

2. DESCRIPTION OF THE ALGORITHM

Let \{x_n\} be the signal of interest and \{w_n\}, the observed signal, given by \(y_n = x_n + w_n\) where \{w_n\} is the additive
noise. We do not assume any knowledge regarding the probability distribution of \(w_n\) except that \(w_n\) is zero mean.
Thus, non-stationary noise assumption is permissible. Next, let \(c_j^k = y_k\) and \(s_j^k = h_k \) be the \(k\)th
wavelet coefficient at resolution \(j, j = 1, \ldots, J\) obtained by passing \(y_n\) through the
subband filtering structure associated with an orthonormal wavelet.
At the lowest resolution \(J\), let the sub-sampled output of the last low-pass filter be denoted by \(s_J^k\).

We then apply the following filter to \(\{c_j^k\} , j = 1, \ldots, J \) and \(\{s_j^k\} \):

\[
\hat{c}_j^k = h_j^k c_j^k + h_{j+1}^k c_{j+1}^k + \lambda^j \sum_{k} (\hat{c}_j^k - \hat{c}_{j-1}^k)^2
\]

\[
\hat{s}_j^k = H_k^j s_j^k + H_{j+1}^k s_{j+1}^k \sum_{k} L_j^k \]

The de-noised version of the signal is obtained by reconstruction from the coefficients \(\hat{c}_j^k\) and \(\hat{s}_j^k\) instead of the
original wavelet coefficients \(c_j^k\) and \(s_j^k\). Here \(\{L_{j+1}^k\}\) are gradient-based switches, defined for each \(j = 1, \ldots, J\)
by

\[
L_{j+1}^k = \begin{cases} 0 & \text{if } ||c_j^k||^2 + ||c_{j+1}^k||^2 > \lambda^j \\ 1 & \text{otherwise} \end{cases}
\]

\(\lambda^j\) are some thresholds to be selected. The switches \(\{L_{j+1}^k\}\) are defined analogous to (3), and have a corresponding
threshold \(T^j\). The role of the switches is to avoid destroying of signal features by recognizing large changes in the
wavelet coefficient amplitude and keeping them unaffected by the smoothing.

We have developed a filtering scheme in the wavelet-domain, using the entire wavelet transform, and avoiding
manual selection of coefficients. We apply selective smoothing at each scale of the transform, and then reconstruct
the de-noised version of the signal. The amount of smoothing is controlled by a regularization factor, and the smoothing
is made selective using gradient-based switches so that important signal features are not destroyed. The selective
smoothing at each scale enables us to obtain good de-noising even on a signal that has both smooth variations as well as
sharp steps and Brownian texture. Simulation results are presented to illustrate the efficiency of our method.

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with respect to \( \{h_k\} \), and likewise, minimizing

\[
B^j = \lambda^j \sum_k (s_k^j - \hat{s}_k^j)^2 + \sum_k \hat{h}_{-1,k}^j (s_k^j - \hat{s}_{k-1}^j)^2
\]

(6)

with respect to \( \{H_k^j\} \). This was done by imposing \( \frac{\partial B^j}{\partial h_k} = 0 \)
and \( \frac{\partial B^j}{\partial h_k^j} = 0 \) for all \( k \). This gives rise to a banded system of
equations in \( \{h_k\} \) at each scale \( j \), and likewise in \( \{H_k^j\} \).

3. HOW THE METHOD WORKS

The wavelet transform can be observed to achieve an effect
similar to de-correlation and energy compaction, so that a
typical signal would often have a transform in which the
essential features of the signal are captured in a single coef-
cient that is markedly different in value from its neighbors
[8]. So, a sudden change in the wavelet coefficient magnitude
is likely to be a signal feature and not a creation of the noise.
Hence performing the blurring across such a sudden change
would be distorting the signal features. So the switches are
used in (1), (2) and (4) to prevent the blurring in these
cases. The cost function (4) indicates that one is trying to
smooth the signal \( \{c_k^j\} \) at each scale \( j \) by reducing the power
in its gradient. The amount of smoothing is controlled by
the regularization parameter. Separate processing at each
scale leads to finer control of the overall performance.

4. RESULTS FROM IMPLEMENTATION

Figure 1 shows the signal on which the scheme described
above was tested. Note that it is similar to the one used by
Mallat and Zhong [5] having isolated singularities as well
as Brownian texture. Figure 2 shows the signal with zero
mean Gaussian noise added to it. The algorithm described
above was run on the noisy signal in Figure 2, using the
Daubechies 6-tap wavelet filter, and the parameters \( t^j \)
and \( \lambda^j \) as shown in Table 1.

\[
\begin{array}{cccccccc}
\hline
j & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
t^j & 1.3 & 1.0 & 0.9 & 1.6 & 2 & 2 & 0 & 0 \\
\hline
\lambda^j & 0.00001 & 0.01 & 0.03 & 5 & 20 & 40 & 10 & 10 \\
\hline
\end{array}
\]

Table 1: Parameters for simulation

with Gaussian noise.

Figure 3 shows the de-noised signal. Note that our algo-
rithm is able to capture both the initial parts of the signal
(consisting of smoother portions with some jumps in be-
 tween) as well as the later portions with Brownian tex-
ture. Thus, the controlled smoothing that our algorithm
gives enables it to perform better than any conventional fil-
tering operation. The SNR improvement from 6db to 12.9db
is substantially better than that obtained by the de-noising
algorithm of [5]-[6], ([5] obtains improvement from 6db to
12.1db); but more importantly the reconstruction follows
the transients in the noiseless signal better than in [5], espe-
cially in the portion with Brownian texture. Figure 4 shows
the signal of Figure 1 with added noise consisting of seg-
ments of different distributions uniform (samples 1 to 50),
gamma (with gamma-parameter 3, samples 51 to 100), ex-
ponential (samples 101 to 180) and Gaussian (samples 181

to 256). The algorithm was run with the Daubechies 6-tap
wavelet filter, and the parameters \( t^j \) and \( \lambda^j \) as shown in
Table 2.

\[
\begin{array}{cccccccc}
\hline
j & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
t^j & 2 & 2 & 5 & 2 & 2 & 1.2 & 0 & 0 \\
\hline
\lambda^j & 0.00001 & 0.0001 & 2 & 30 & 1000 & 1000 & 10 & 10 \\
\hline
\end{array}
\]

Table 2: Parameters for simulation

with non-stationary noise.

Figure 5 shows the output of the algorithm in this case.
Again we see that both the smoother parts as well as the later
irregular portions of the input signal have been captured by
the algorithm.

5. CONCLUSION

We conclude that filtering in the wavelet domain, using
smoothing without regularization and with the gradient-based
switches, is able to recover both smooth and irregular parts
of the input signal from the noisy signal. Moreover the algo-
rithm is able to handle non-stationary or variable distribu-
tion noise.

REFERENCES

composition: The Wavelet Representation," IEEE
Class of Transient Detection Algorithms - A Unified
Processing with Wavelets," IEEE Trans. Inform. Theo-
from Multiscale Edges," IEEE Trans. PAMI, Vol. 14,
[7] O. Bertrand, J. Bohorquez and J. Pernier, "Time-
Frequency Digital Filtering Based on an Invertible
Wavelet Transform: An Application to Evoked Poten-
[8] D.L. Donoho and I.M. Johnstone, "Ideal spatial adap-
p. 1279-1284, May 1996.
Figure 1. Input signal, free of noise.

Figure 2. Input signal with Gaussian noise added, SNR=6db.

Figure 3. De-noised signal obtained from our algorithm, SNR=12.9db.
Figure 4. Input signal with non-stationary noise added, SNR=6db.

Figure 5. De-noised signal obtained from our algorithm, SNR=10.5db.