LOCALIZED SUBCLASSES OF QUADRATIC TIME-FREQUENCY REPRESENTATIONS

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ABSTRACT
We discuss the existence of classes of quadratic time-frequency representations (QTFRs), e.g. Cohen, power, and generalized time-shift covariant, that satisfy a time-frequency (TF) concentration property. This important property yields perfect QTFR concentration along group delay curves. It also (1) simplifies the QTFR formulation and property kernel constraints as the kernel reduces from 2-D to 1-D, (2) reduces the QTFR computational complexity, and (3) yields simplified design algorithms. We derive the intersection of Cohen’s class with the new power exponential class, and show that it belongs to Cohen’s localized-kernel subclass. In addition to the TF shift covariance and concentration properties, these intersection QTFRs preserve power exponential time shifts, important for analyzing signals passing through exponentially dispersive systems.

1. INTRODUCTION
Quadratic time-frequency representations (QTFRs) can be grouped into classes based upon covariance properties that they satisfy. Cohen’s class QTFRs [1, 2, 3] are covariant to constant time-frequency (TF) shifts, affine class QTFRs [4, 2, 3] are covariant to dilations and constant time shifts, and the hyperbolic class [5, 6] and power classes QTFRs [7, 5] are covariant to dilations and dispersive time shifts.

In this paper, we focus on subclasses of QTFRs whose kernels have a localized structure. Such QTFRs satisfy a TF concentration property, an important property that yields perfect concentration of the QTFRs along specified group delay curves for certain types of signals. This concentration property of a QTFR $T_X(t, f)$ of a signal $x_c(t)$ (with Fourier transform $X_c(f)$) is given by

$$X_c(f) = r(f)e^{-j2\pi \alpha \lambda(f)} \Rightarrow T_X(t, f) = r^2(f)\delta(t-c\mu(f)). \quad (1)$$

Thus, given the signal $X_c(f)$ with amplitude function $r(f)$ $> 0$ and one-to-one phase function $\alpha(f) \in \mathbb{R}$, we want the QTFR to be perfectly concentrated along the signal’s group delay $c\mu(f) = c\frac{d}{df} \lambda(f)$. Here, $f_0 > 0$ is a fixed reference frequency. This property results in a simplification of the signal-independent 2-D kernel to a 1-D localized-kernel characterizing the QTFRs. The localized-kernel idea was introduced by the Bertrands based upon tomography [8], and then extended by others to the affine localized-kernel subclass [4, 9, 10], and to the hyperbolic localized-kernel subclass [6, 5]. In this paper, we investigate the subclasses of QTFRs that satisfy the TF concentration property in (1) for other known QTFR classes such as Cohen’s class [1, 2, 3] and the power classes [7, 5].

In addition, we consider classes of QTFRs that satisfy the generalized time-shift covariance property defined as

$$(D^{\xi}_\nu X)(f) = e^{-j2\pi \nu \xi(f)} X(f) \Rightarrow T_{D^{\xi}_\nu X}(t, f) = T_X(t-c\tau(f), f) \quad (2)$$

with $\tau(f) = -\frac{d}{df} \xi(f)$ and $\xi(b)$ a one-to-one function [11, 12]. We derive the concentration property kernel constraints for these generalized time-shift covariant classes, thus providing a unifying framework for many localized-kernel QTFRs.

2. COHEN’S LOCALIZED-KERNEL QTFRs
Any Cohen’s class [1] QTFR, $T^{(C)}_\nu(t, f)$, with signal-independent kernel, satisfies the time-shift covariance property given by (2) with $\xi(b) = b$ and $\tau(f) = 1/f_0$, and the frequency-shift covariance property defined as

$$(M_X X)(f) = X(f-\nu) \Rightarrow T^{(C)}_{M_X X}(t, f) = T^{(C)}_X(t, f-\nu). \quad (3)$$

These shift covariances are important in applications where the TF structure is analyzed with constant TF resolution such as in speech. Based on the TF shift covariance properties, any member of Cohen’s class can be expressed in terms of a 2-D kernel $\Phi^{(C)}_\nu(\hat{f}, \nu)$ [2, 3]:

$$T^{(C)}_X(t, f) = \int \int \Phi^{(C)}_\nu(\hat{f} - \hat{f}, \nu) U_X(\hat{f}, \nu) e^{j2\pi \nu t} d\hat{f} d\nu \quad (4)$$

where $U_X(\hat{f}, \nu) = X(\hat{f} + \nu/2) X^*(\hat{f} - \nu/2)$. Two interesting Cohen’s class QTFRs are the Wigner distribution (WD) and Cohen’s counterpart of the $k$th power Bertrand distribution [5, CBDs, $k \neq 0$]. These QTFRs are defined as in (4) with well-localized kernels given by $\Phi^{(C)}_{WD}(\hat{f}, \nu) = \delta(\hat{f})$ and $\Phi^{(C)}_{CBD_k}(\hat{f}, \nu) = \delta(\hat{f} - \frac{k}{2}\nu) \sin \left(\frac{\pi}{2}\frac{\nu}{k}\right)$; they result in highly concentrated representations for linear and exponential chirps, respectively.

We propose Cohen’s localized-kernel subclass (LCC) as the subclass of QTFRs consisting of all Cohen’s class QTFRs whose kernel $\Phi^{(C)}_\nu(\hat{f}, \nu)$ (in (4)) is perfectly localized along a curve $\hat{f} = F^{(C)}_X(\nu)$ in the $(\hat{f}, \nu)$-plane, i.e.,

$$\Phi^{(C)}_\nu(\hat{f}, \nu) = G^{(C)}_X(\nu) \delta(\hat{f} - F^{(C)}_X(\nu)). \quad (5)$$

The 2-D kernel of each LCC QTFR is reduced to two, 1-D functions, $F^{(C)}_X(\nu)$ and $G^{(C)}_X(\nu)$, with $G^{(C)}_X(\nu) \geq 0$, that uniquely characterize the QTFR. This greatly simplifies the QTFR analysis as it reduces the LCC concentration in (4) to

$$(T^{(C)}_X(t, f) = \int F^{(C)}_X(\nu) U_X(f - F^{(C)}_X(\nu), \nu) e^{j2\pi \nu t} d\nu \quad (6)$$

from two integrals to just one integral. Furthermore, it simplifies the kernel constraints for desirable QTFR properties [2].
The LCC is also important as it contains all Cohen’s class QTFRs that satisfy the TF concentration property in (1). In particular, we show next that the localized-kernel structure in (5) is necessary for the corresponding QTFR to satisfy the TF concentration property along group delay curves in (1) [9]. For a given \( r(f) \) and \( \lambda(b) \) in (1), and if \( \Lambda_0^{(r)}(\hat{f}) \triangleq \lambda(f-\hat{f}+\nu/2)/f_2 - \lambda(f-\hat{f}-\nu/2)/f_2 \) is assumed to be one-to-one and differentiable for fixed \( f, \nu \), then (1) is satisfied by a Cohen’s class QTFR if and only if the following three conditions are satisfied:

C-I There exists a real function \( F^{(C)}(\nu) \), independent of \( f \), that satisfies \( \Lambda_0^{(r)}(F^{(C)}(\nu)) = \mu_f(f) \) for all \( f, \nu \in \mathbb{R} \).

C-II The ratio \( r^2(f)/R^{(C)}(f, \nu) \) is independent of \( f \), where

\[
R^{(C)}(f, \nu) = \begin{cases} r(f-\hat{f}+\nu/2)/f_2 - r(f-\hat{f}-\nu/2)/f_2 & \text{if } f \geq 0, \\ r(\hat{f}) & \text{if } f < 0. 
\end{cases}
\]

C-III The kernel \( \Phi^{(C)}(f, \nu) \) of the LCC is given by (5) with

\[
G^{(C)}(\nu) = r^2(f)/R^{(C)}(f, \nu). \quad (28)
\]

In particular, fixing \( r(f) = e^{-\frac{f^2}{2\tau_f^2}}, \kappa \neq 0 \), condition C-II always holds with \( G^{(C)}(\nu) = e^{-\frac{\nu^2}{2\tau^2}} \). For example, if \( \lambda(b) = b^\alpha \) in property (1), condition C-II is satisfied for arbitrary \( F^{(C)}(\nu) \). If \( \lambda(b) = e^{\omega(b)} \), then the TF concentration property for power exponential chirps \( \chi_n(f) \) in (1) is satisfied by Cohen’s class QTFRs with \( F^{(C)}(\nu) = e^{\frac{\nu^2}{2\tau^2}} \ln \left[ \sinh \left( \frac{\nu}{\tau} \right) / \frac{\nu}{\tau} \right] \) and \( G^{(C)}(\nu) = e^{-\frac{\nu^2}{2\tau^2}} \). Some LCC QTFRs include the Wigner distribution with 1-D kernels in (5) given by \( G^{(W/D)} = 1 \) and \( F^{(W/D)}(\nu) = 0 \) and the CBD with 1-D kernels in (5) given by \( G^{(CBD)} = 1 \) and \( F^{(CBD)}(\nu) = e^{\frac{\nu^2}{2\tau^2}} \ln \left[ \sinh \left( \frac{\nu}{\tau} \right) / \frac{\nu}{\tau} \right] \).

Proposition 1: Under the restrictions on the kernels, a Cohen’s class QTFR \( \Phi^{(C)}(f, \nu) \) satisfies the TF concentration property in (1) if and only if

\[
\exists \Psi^{(C)}(f, \nu) = F^{(C)}(\nu), \quad \text{satisfies condition C-II,}
\]

2.1. Intersection with Power Exponential Classes

The 1-D kernel function \( G^{(C)}(\nu) \) of the CBD, \( \kappa \neq 0 \), satisfies two covariance properties: the frequency-shift covariance in (3), and the power exponential time-shift covariance (defined as in (2)) with \( \xi(b) = e^{-\omega(b)} \).

The 1-D kernels in (5) given by \( G^{(CBD)} = 1 \) and \( F^{(CBD)}(\nu) = 0 \) are interesting as they are the only cases where the 1-D kernels in (5) are SOTF. The 1-D kernels in (5) given by \( G^{(CBD)} = 1 \) and \( F^{(CBD)}(\nu) = 0 \) are interesting as they are the only cases where the 1-D kernels in (5) are SOTF. The 1-D kernels in (5) given by \( G^{(CBD)} = 1 \) and \( F^{(CBD)}(\nu) = 0 \) are interesting as they are the only cases where the 1-D kernels in (5) are SOTF.
If given a given arbitrary but to satisfy the TF concentration property in (1). Here, $\Xi_{\beta}^{(G)}(b) = \frac{x}{2} \coth \left( \frac{x}{2} \right)$. For $\xi(b) = b$, the GC in (9) is Cohen’s class in (4), and for $\xi(b) = b$, the GC yields the hyperbolic class [5]. Any generalized warped affine class (GA), $T_X^{(GA)}(t, f)$, is obtained by warping the affine class [12, 14], and can be written in terms of a 2-D kernel $\Phi_T^{(G)}(b, \beta)$ as

\[
T_X^{(GC)}(t, f) = \int \Phi_T^{(GC)}(b, \beta) \frac{b}{\xi} V_X(b, \beta) e^{j2\pi r T / \beta} \, db \, d\beta.
\] (10)

The choice of $\xi(b)$ determines the class to which the GA in (10) simplifies. For $\xi(b) = b$, the GA is the affine class, for $\xi(b) = \xi(b)$ it is the nth power class in (7), and for $\xi(b) = e^{b \xi(b)}$ it is the nth power exponential class given in Section 2.1.

Here, we propose a localized-kernel subclass of the GC and a localized-kernel subclass of the GA. Specifically, we want to group together generalized QTFRs that satisfy the TF concentration property in (1), and provide perfect concentration of the QTFR along given group delay curves. Such QTFRs will necessarily have a kernel with a localized structure as we will show next. For example, the localized GC consists of all generalized warped Cohen’s QTFRs whose 2-D kernel in (9) is perfectly localized along a curve $b = T_T^{(GC)}(\beta)$ in the $(b, \beta)$-plane,

\[
\Phi_T^{(GC)}(b, \beta) = G_T^{(GC)}(\beta) \delta(b - F_T^{(GC)}(\beta)),
\] (11)

where $F_T^{(GC)}(\beta) \in \mathbb{R}$ and $G_T^{(GC)}(\beta) \geq 0$ are 1-D functions that characterize $T_T^{(GC)}$. All localized GA QTFRs in (10) have the same localized-kernel structure as in (11), but with $\Phi_T^{(GC)}$ replaced by $G_T^{(GC)}$. GC and GA QTFRs satisfy the TF concentration property (1) if and only if the conditions listed in Table 1 hold. For example, to determine whether a GC QTFR satisfies property (1) for a given $\xi(b)$, we first need to check whether a frequency-independent kernel, $F_T^{(GC)}(\beta)$, exists that satisfies condition G-I in Table 1. If $F_T^{(GC)}(\beta)$ does not exist, then there does not exist any GC QTFR satisfying (1). If a kernel $F_T^{(GC)}(\beta)$ satisfying condition G-I exists, then we substitute it in the frequency-independent ratio condition in G-II to form $G_T^{(GC)}(\beta)$ for a given $r(f)$. Lastly, condition G-III yields the localized-kernel structure in (11).

It is important to note that when the analyzing signal’s phase function $\lambda(b)$ in (1) equals the function $\xi(b)$ that specifies the generalized time-shift covariant class in GC (9) (resp. GA (10)), condition G-I always holds for the localized GC (resp. localized GA) in Table 1. Specifically, when $\lambda(b) = \xi(b)$ in Table 1, then $\Lambda_{\beta}^{(G)}(F_T^{(GC)}(\beta) = \beta$ for the GC and $\Lambda_{\beta}^{(G)}(F_T^{(GA)}(\beta) = \xi(b)$ for the GA, and

<table>
<thead>
<tr>
<th>Class</th>
<th>$A_{\lambda}^{(class)}(b)$</th>
<th>$R_{\lambda}^{(class)}(b)$</th>
<th>$r(\xi^{(class)}(b))$</th>
<th>$\Lambda_{\beta}^{(class)}(F_T^{(class)}(\beta))$</th>
<th>$\Phi_T^{(class)}(b, \beta) = G_T^{(class)}(\beta) \delta(b - F_T^{(class)}(\beta))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GC or GA</td>
<td>$\lambda(b)$</td>
<td>$\xi(b)$</td>
<td>$r(\xi(b))$</td>
<td>$\xi^{(class)}(b)$</td>
<td>$G_T^{(class)}(\beta) = \xi^{(class)}(b)$</td>
</tr>
</tbody>
</table>

Table 1: Necessary conditions for generalized warped Cohen’s (class = GC) or generalized warped affine (class = GA) QTFRs to satisfy the TF concentration property in (1). Here, $\Xi_{\beta}^{(G)}(b) = \frac{x}{2} \coth \left( \frac{x}{2} \right)$, and $\Xi^{(G)}(b) = \xi^{(class)}(b)$ where $\xi(b)$ is given in (9) and (10). Also, $\Phi_T^{(class)}(b, \beta) = G_T^{(class)}(\beta) \delta(b - F_T^{(class)}(\beta))$.

4. GENERALIZED WARPED LOCALIZED KERNEL QTFRs

Generalized time-shift covariant QTFRs satisfy the generalized time-shift covariation property in (2) for given $\xi(b)$ [11, 12]. These QTFRs are especially useful when the shift is matched to group delay functions $\tau(f)$, and they are important for analyzing signals passing through systems with matched group delay characteristics. Two QTFR classes that satisfy (2) are the generalized warped Cohen’s class and the generalized warped affine class [11, 12, 14]. These two classes depend on the choice of $\xi(b)$ in (2) that defines the type of warping applied to either Cohen’s class or to the affine class. Thus, the generalized warped classes provide a unifying framework for Cohen’s, affine, hyperbolic, power, exponential, and power exponential, classes [12].

QTFRs of the generalized warped Cohen’s class (GC), $T_X^{(GC)}(t, f)$, are obtained by warping Cohen’s class QTFRs [12, 14] using a mapping specified by $\xi(b)$. Any GC QTFR can be expressed in terms of a 2-D kernel $\Phi_T^{(GC)}(b, \beta)$ as

\[
T_X^{(GC)}(t, f) = \int \Phi_T^{(GC)}(\xi(b) t, \beta) V_X(b, \beta) e^{j2\pi r T / \beta} \, db \, d\beta.
\] (9)
condition G-I holds for arbitrary $F_1^{(GC)}(\beta)$ and $F_2^{(GA)}(\beta)$ kernels. This is further demonstrated in Table 2 where we computed the $G_r(\beta)$ and $F_r(\beta)$ kernels for various choices of signal parameters $r(f)$ and $\lambda(b)$ in (1), and for various choices of time shift parameter $\xi(b)$ in (9) and (10). For example, when $\xi(b) = \xi_2(b) = \text{sgn}(b)|b|^a$ in (7), the localized GA is the $a$th power localized kernel subclass in Section 3. From Table 2, if $r(f) = r_0(|f|) = F^{(\text{PE})}(|f|) - \frac{\alpha_0}{2} \text{coth} \left( \frac{\alpha_0}{2} \right)$ and $\lambda(b) = \text{sgn}(b)$ such that $X_c(f) = F^{(\text{PE})}(|f|)$ in (1) is a hyperbolic chirp, then the kernels simplify to the kernels of the power Bertrand distribution, i.e. $F_1^{(GA)}(\beta) = F_2^{(\text{PE})}(\beta) = -\frac{\alpha_0}{2} \text{coth} \left( \frac{\alpha_0}{2} \right)$ and $C_2^{(GA)}(\beta) = C_2^{(\text{PE})}(\beta) = \frac{\alpha_0}{2} \text{coth} \left( \frac{\alpha_0}{2} \right)$. Thus, the generalization simplifies to known results which state that the power Bertrand distribution is perfectly concentrated along hyperbolic in the TF plane for hyperbolic chirps [5].

The localized-kernel structure in Table 2 is also important as it simplifies the formulation and property kernel constraints for a GA or QTFR [13], as its 2-D kernel is now in terms of two 1-D kernels. The localized GC and the localized GA are also important as they provide a unifying framework for many localized-kernel QTFR subclasses. From Tables 1 and 2, the localized GC simplifies to Cohen's localized-kernel subclass in Section 2 when $\xi(b) = b$ and $G^{(GC)}(\beta) = G_1^{(\text{PE})}(\beta)$, $F_1^{(\text{PE})}(\beta) = F_2^{(\text{PE})}(\beta)$, $F_{2}^{\{\text{class}}(\beta)\} = F_{2}^{\{\text{class}}(\beta)\} = F_{2}^{\{\text{class}}(\beta)\} = F_{2}^{\{\text{class}}(\beta)\}$, $R_{2}^{(\text{class}}(\beta)\} = R_{2}^{(\text{class}}(\beta)\}$, and $R_{2}^{(\text{class}}(\beta)\} = R_{2}^{(\text{class}}(\beta)\}$, and $R_{2}^{(\text{class}}(\beta)\} = R_{2}^{(\text{class}}(\beta)\}$. When $\lambda(b) = b$, the localized GC simplifies to the hyperbolic localized-kernel subclass [6]. Similarly, the localized GA simplifies to the affine localized-kernel subclass [9, 10] when $\xi(b) = b$, to the $a$th power localized-kernel subclass in Section 3 when $\xi(b) = \xi_2(b)$, and to the exponential localized-kernel subclass [13] when $\xi(b) = e^b$.

5. CONCLUSION

The TF concentration property in (1) is an important property as it provides perfectly concentrated analysis of a signal along its group delay function. We showed that when constraining a QTFR to satisfy this property, the QTFR’s kernel must necessarily have a localized structure, and we derived this kernel structure for known classes of QTFRs such as Cohen’s class and the power classes. We further generalized the localized-kernel subclasses to provide a unifying framework for all generalized-time shift covariant QTFR classes including Cohen’s class, the affine class, the hyperbolic class, the $a$th power class, the exponential class, and the $a$th power exponential class. As the localized-kernel structure reduces the 2-D kernel of the various classes to two, 1-D kernels, the localized-kernel subclasses have simplified formulations and kernel constraints. The structure can also reduce the QTFR computational complexity and provide more intuitive QTFR design algorithms.

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6. REFERENCES